Supplement to 'Variance estimation in the particle filter'

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S1 Notation and conventions for the genealogical tracing variables and measures on $\mathcal{X}^{\otimes 2}$

We first write an expression for the probability mass function of the genealogical random variables (K^1, K^2) described in Section 3.1. Let $[N_{0:n}] = [N_0] \times \cdots \times [N_n]$. Then with $a = (a_0, \ldots, a_{n-1}) \in [N_0]^{N_1} \times \cdots \times [N_{n-1}]^{N_n}$, $z = (z_0, \ldots, z_n) \in \mathsf{X}^{N_0} \times \cdots \times \mathsf{X}^{N_n}$ and $k^1 = (k_0^1, \ldots, k_n^1) \in [N_{0:n}]$, we define

$$C_1(a, z; k^1) = \frac{\mathbb{I}(k_n^1 \in [N_n])}{N_n} \prod_{p=1}^{n} \mathbb{I}(k_{p-1}^1 = a_{p-1}^{k_p^1}).$$

With $k^2 = (k_0^2, \ldots, k_n^2) \in [N_{0:n}]$, we also define

$$C_2(a, z, k^1; k^2) = \frac{\mathbb{I}(k_n^2 \in [N_n])}{N_n} \prod_{p=1}^{n} \left\{ \mathbb{I}\left(k_p^2 \neq k_p^1, k_{p-1}^2 = a_{p-1}^{k_p^2}\right) + \frac{\mathbb{I}\left(k_p^2 = k_p^1\right) G_{p-1}(z_{p-1}^{k_{p-1}^2})}{\sum_{j=1}^{N_{p-1}} G_{p-1}(z_{p-1}^j)} \right\}.$$

Note that with a, z fixed $C_1(a, z; \cdot)$ is a probability mass function on $[N_{0:n}]$, as is $C_2(a, z, k^1; \cdot)$ when (a, z, k^1) is fixed. With C_1 and C_2 so-defined and

$$C(A, \zeta; k^{1:2}) = C_1(A, \zeta; k^1) C_2(A, \zeta, k^1; k^2), \tag{S1}$$

it is evident that $C(A, \zeta; \cdot)$ is the probability mass function of (K^1, K^2).

We now recursively define a collection of measures on $\mathcal{X}^{\otimes 2}$, which include the measure μ_b of Section 3.2. We define $\tilde{G}_p = G_p^{\otimes 2}$, $p \in \{0, \ldots, n\}$. For any $b \in B_n$ and writing $x_p^{1:2} = (x_p^1, x_p^2)$, we define

$$\tilde{M}_0^{b_0}(\mathrm{d}x_0^{1:2}) = M_0(\mathrm{d}x_0^1) \left[\mathbb{I}(b_0 = 0) M_0(\mathrm{d}x_0^2) + \mathbb{I}(b_0 = 1) \delta_{x_0^1}(\mathrm{d}x_0^2)\right],$$

and for each $p \in \{1, \ldots, n\}$,

$$\tilde{M}_p^{b_p}(x_{p-1}^{1:2}, \mathrm{d}x_p^{1:2}) = M_p(x_{p-1}^1, \mathrm{d}x_p^1) \left[\mathbb{I}(b_p = 0) M_p(x_{p-1}^2, \mathrm{d}x_p^2) + \mathbb{I}(b_p = 1) \delta_{x_p^1}(\mathrm{d}x_p^2)\right].$$

We now define, similarly to (1), $\mu_{b_0} = \tilde{M}_0^{b_0}$ and for $p \in \{1, \ldots, n\}$, recursively,

$$\mu_{b_{0:p}}(S) = \int_{\mathsf{X}^2} \mu_{b_{0:p-1}}(\mathrm{d}x_{p-1}^{1:2}) \tilde{G}_{p-1}(x_{p-1}^{1:2}) \tilde{M}_p^{b_p}(x_{p-1}^{1:2}, S), \quad S \in \mathcal{X}^{\otimes 2}. \tag{S2}$$

It follows that for $b \in B_n$, the measure μ_b described is defined by (S2).

S2 Algorithms for computing the estimates

Algorithm 2 provides pseudo-code for computing $V_n^N(\varphi)$ in $\mathcal{O}(N)$ time after Algorithm 1 has been run, with Lemma 6 providing its justification.

Algorithm 2. *Computing $V_n^N(\varphi)$.*

1. *Let $S_{0,n}$ be an array of length N_0, initialized to* 0.
2. *For $j \in [N_n]$, set $S_{0,n}[E_n^j] \leftarrow S_{0,n}[E_n^j] + \varphi(\zeta_n^j)/N_n$.*
3. *Set*
$$m_\star^N(\varphi) \leftarrow \left(\prod_{p=0}^{n} \frac{N_p}{N_p - 1} \right) \left\{ \eta_n^N(\varphi)^2 - \sum_{i \in [N_0]} S_{0,n}[i]^2 \right\}.$$
4. *Set $V_n^N(\varphi) \leftarrow \eta_n^N(\varphi)^2 - m_\star^N(\varphi)$.*

Lemma 6. *For any $\varphi \in \mathcal{L}(\varphi)$, $m_\star^N(\varphi) = \mu_{0_n}^N(\varphi^{\otimes 2})/\gamma_n^N(1)^2$.*

Proof. We have $S_{0,n}[i] = N_n^{-1} \sum_{j \in [N_n] : E_n^j = i} \varphi(\zeta_n^j)$, for each $i \in [N_0]$ upon completion of step 2 of Algorithm 2. Noting that

$$\frac{1}{N_n^2} \sum_{i,j : E_n^i \neq E_n^j} \varphi(\zeta_n^i)\varphi(\zeta_n^j) = \eta_n^N(\varphi)^2 - \sum_{i \in [N_0]} \left\{ \frac{1}{N_n} \sum_{j : E_n^j = i} \varphi(\zeta_n^j) \right\}^2,$$

the result follows from (14) and (16). □

Algorithm 3 provides pseudo-code for computing each $v_{p,n}^N(\varphi)$, $p \in \{0, \ldots, n\}$ in $\mathcal{O}(Nn)$ time after Algorithm 1 has been run, with Lemma 7 providing its justification. The computation involves indexing sets of offspring indices, which we define as

$$O_p^i = \{j \in [N_{p+1}] : A_p^j = i\}, \qquad p \in \{0, \ldots, n-1\}, \quad i \in [N_p]. \tag{S3}$$

Algorithm 3. *Computing each $v_{p,n}^N(\varphi)$, and $v_n^N(\varphi)$.*

1. *Let $S_{n,n}$ be an array of length N_n, such that $S_{n,n}[i] = \varphi(\zeta_n^i)/N_n$.*
2. *For each $p = n-1, \ldots, 0$:*
   (a) *Let $S_{p,n}$ be a zero array of length N_p.*
   (b) *For $j \in [N_{p+1}]$, set $S_{p,n}[A_p^j] \leftarrow S_{p,n}[A_p^j] + S_{p+1,n}[j]$.*
3. *For each $p \in \{0, \ldots, n-1\}$:*
   (a) *Let t_p be an array of length N_p such that $t_p[i] = G_p(\zeta_p^i) / \sum_{j \in [N_p]} G_p(\zeta_p^j)$.*
   (b) *Let $g_{0,p}$ be a zero array of length N_0. For $j \in [N_p]$, set $g_{0,p}[E_p^j] \leftarrow g_{0,p}[E_p^j] + t_p[j]$.*
4. *Set $m_\star^N(\varphi) \leftarrow \left(\prod_{p=0}^{n} \frac{N_p}{N_p-1} \right) \left\{ \eta_n^N(\varphi)^2 - \sum_{i \in [N_0]} S_{0,n}[i]^2 \right\}$.*
5. *Set $m_{n,n}^N(\varphi) \leftarrow (N_n - 1) \left[\prod_{q=0}^{n} \frac{N_q}{N_q-1} \right] \sum_{i \in [N_n]} S_{n,n}[i]^2 \left(1 - g_{0,n-1}[E_n^i]\right)$.*
6. *For $p \in \{0, \ldots, n-1\}$,*
   (a) *Let $R_p^{(1)}$ and $R_p^{(2)}$ be zero arrays of length N_p.*
   (b) *For $j \in [N_{p+1}]$, set $R_p^{(1)}[A_p^j] \leftarrow R_p^{(1)}[A_p^j] + S_{p+1,n}[j]$.*
   (c) *For $j \in [N_{p+1}]$, set $R_p^{(2)}[A_p^j] \leftarrow R_p^{(2)}[A_p^j] + S_{p+1,n}[j]^2$.*
   (d) *If $p \geq 1$, set $m_{p,n}^N(\varphi) \leftarrow (N_p - 1) \left(\prod_{q=0}^{n} \frac{N_q}{N_q-1} \right) \sum_{i \in [N_p]} \left(R_p^{(1)}[i]^2 - R_p^{(2)}[i] \right) \left(1 - g_{0,p-1}[E_p^i]\right)$; otherwise set $m_{0,n}^N(\varphi) \leftarrow (N_0 - 1) \left(\prod_{q=0}^{n} \frac{N_q}{N_q-1} \right) \sum_{i \in [N_0]} \left(R_0^{(1)}[i]^2 - R_0^{(2)}[i] \right)$.*
7. *For $p \in \{0, \ldots, n\}$, set $v_{p,n}^N(\varphi) \leftarrow m_{p,n}^N(\varphi) - m_\star^N(\varphi)$. Set $v_n^N(\varphi) \leftarrow \sum_{p=0}^{n} c_p^{-1} v_{p,n}^N(\varphi)$.*

Lemma 7. *For any* $\varphi \in \mathcal{L}(\mathcal{X})$ *and* $p \in \{0, \dots, n\}$, $m_{p,n}^N(\varphi) = \mu_{e_p}^N(\varphi^{\otimes 2})/\gamma_n^N(1)^2$.

Proof. We define, for any $p \in \{0, \dots, n-1\}$,

$$\bar{G}_{0,p}^i := \frac{\sum_{j \in [N_p]: E_p^j = i} G_p(\zeta_p^j)}{\sum_{j \in [N_p]} G_p(\zeta_p^j)}, \qquad i \in [N_0].$$

We also define $\psi_{n,n}^i(\varphi) = \varphi(\zeta_n^i)/N_n$ for $i \in [N_n]$, and for $p \in \{0, \dots, n-1\}$,

$$\psi_{p,n}^i(\varphi) = \sum_{j \in [N_{p+1}]: A_p^j = i} \psi_{p+1,n}^j(\varphi), \qquad i \in [N_p].$$

In Algorithm 3, $S_{p,n}[i] = \psi_{p,n}^i(\varphi)$ for each $i \in [N_p]$ and each $p \in \{0, \dots, n\}$ upon completion of step 2. Upon completion of step 3, $g_{0,p}[i] = \bar{G}_{0,p}^i$ for each $i \in [N_0]$ and $p \in \{0, \dots, n-1\}$. Finally, upon completion of step 6(c), $R_p^{(1)}[i] = \sum_{j \in O_p^i} \psi_{p+1,n}^i$ and $R_p^{(2)}[i] = \sum_{j \in O_p^i} \left(\psi_{p+1,n}^i\right)^2$ for each $i \in [N_p]$ and $p \in \{0, \dots, n-1\}$. We now verify that $m_{p,n}^N(\varphi) = \mu_{e_p}^N(\varphi^{\otimes 2})/\gamma_n^N(1)^2$ for each $p \in \{0, \dots, n\}$.

When $p = n$, we have

$$\frac{\mu_{e_n}^N(\varphi^{\otimes 2})}{\left\{(N_n - 1)\prod_{q=0}^n \frac{N_q}{N_q - 1}\right\}\gamma_n^N(1)^2} = \sum_{k^{1:2} \in \mathcal{I}(e_n)} C(A, \zeta; k^{1:2})\varphi(\zeta_n^{k_n^1})\varphi(\zeta_n^{k_n^2})$$

$$= \sum_{k^{1:2} \in [N_{0:n}]^2} \frac{\mathbb{I}(k_n^1 = k_n^2)}{N_n^2}\varphi(\zeta_n^{k_n^1})^2 \sum_{i \in [N_{n-1}]} \frac{G_{n-1}(\zeta_{n-1}^i)\mathbb{I}(E_{n-1}^i \neq E_n^{k_n^1})}{\sum_{j \in [N_{n-1}]} G_{n-1}(\zeta_{n-1}^j)} = \sum_{i \in [N_n]} \frac{1}{N_n^2}\left(1 - \bar{G}_{0,n-1}^{E_n^i}\right)\varphi(\zeta_n^i)^2,$$

and we conclude by noting that $\psi_{n,n}^i(\varphi) = \varphi(\zeta_n^i)/N_n$. When $p = 0$, we have

$$\frac{\mu_{e_0}^N(\varphi^{\otimes 2})}{\left\{(N_0 - 1)\prod_{q=0}^n \frac{N_q}{N_q - 1}\right\}\gamma_n^N(1)^2} = \sum_{k^{1:2} \in \mathcal{I}(e_0)} C(A, \zeta; k^{1:2})\varphi(\zeta_n^{k_n^1})\varphi(\zeta_n^{k_n^2})$$

$$= \sum_{k^{1:2} \in [N_{0:n}]^2} \frac{\mathbb{I}(k_0^1 = k_0^2)}{N_n^2}\left\{\prod_{q=1}^n \mathbb{I}(k_q^1 \neq k_q^2, k_{q-1}^1 = A_{q-1}^{k_q^1}, k_{q-1}^2 = A_{q-1}^{k_q^2})\right\}\varphi(\zeta_n^{k_n^1})\varphi(\zeta_n^{k_n^2})$$

$$= \sum_{i \in [N_0]} \sum_{j \neq j' \in O_0^i} \psi_{1,n}^j(\varphi)\psi_{1,n}^{j'}(\varphi) = \sum_{i \in [N_0]}\left(\sum_{j \in O_0^i} \psi_{1,n}^j(\varphi)\right)^2 - \sum_{j \in O_0^i} \psi_{1,n}^j(\varphi)^2.$$

Finally when $p \in \{1, \dots, n-1\}$, we have

$$\frac{\mu_{e_p}^N(\varphi^{\otimes 2})}{\left\{(N_p - 1)\prod_{q=0}^n \frac{N_q}{N_q - 1}\right\}\gamma_n^N(1)^2} = \sum_{k^{1:2} \in \mathcal{I}(e_p)} C(A, \zeta; k^{1:2})\varphi(\zeta_n^{k_n^1})\varphi(\zeta_n^{k_n^2})$$

$$= \sum_{k^{1:2} \in [\mathbf{N}]^2} \frac{\mathbb{I}(k_p^1 = k_p^2)}{N_n^2}\left\{\prod_{q=p+1}^n \mathbb{I}(k_q^1 \neq k_q^2, k_{q-1}^1 = A_{q-1}^{k_q^1}, k_{q-1}^2 = A_{q-1}^{k_q^2})\right\}$$

$$\cdot\, \varphi(\zeta_n^{k_n^1})\varphi(\zeta_n^{k_n^2}) \sum_{i \in [N_{p-1}]} \frac{G_{p-1}(\zeta_{p-1}^i)\mathbb{I}(E_{p-1}^i \neq E_p^{k_p^1})}{\sum_{j \in [N_{p-1}]} G_{p-1}(\zeta_{p-1}^j)}$$

$$= \sum_{i \in [N_0]}\left\{\sum_{j \neq j' \in O_p^i} \psi_{p+1,n}^j(\varphi)\psi_{p+1,n}^{j'}(\varphi)\right\}\left(1 - \bar{G}_{0,p-1}^{E_p^i}\right)$$

$$= \sum_{i \in [N_p]}\left(1 - \bar{G}_{0,p-1}^{E_p^i}\right)\left[\left\{\sum_{j \in O_p^i} \psi_{p+1,n}^j(\varphi)\right\}^2 - \sum_{j \in O_p^i} \psi_{p+1,n}^j(\varphi)^2\right].$$

□

S3 Proof of Corollary 1

Proof of Corollary 1. Rearranging (4), for any $\varphi \in \mathcal{L}(\mathcal{X})$,

$$\begin{aligned} V_n^N(\varphi) &= \eta_n^N(\varphi)^2 - \left(\prod_{p=0}^n \frac{N_p}{N_p-1}\right)\frac{1}{N_n^2}\sum_{i,j:E_n^i\neq E_n^j}\varphi(\zeta_n^i)\varphi(\zeta_n^j) \\ &= \eta_n^N(\varphi)^2 - \left(\prod_{p=0}^n \frac{N_p}{N_p-1}\right)\left\{\eta_n^N(\varphi)^2 - \frac{1}{N_n^2}\sum_{i,j:E_n^i= E_n^j}\varphi(\zeta_n^i)\varphi(\zeta_n^j)\right\} \\ &= \left(\prod_{p=0}^n \frac{N_p}{N_p-1}\right)\left[\frac{1}{N_n^2}\sum_{i,j:E_n^i= E_n^j}\varphi(\zeta_n^i)\varphi(\zeta_n^j) - \eta_n^N(\varphi)^2\left\{1-\prod_{p=0}^n\left(1-\frac{1}{N_p}\right)\right\}\right] \\ &= \left(\prod_{p=0}^n \frac{N_p}{N_p-1}\right)\left[\frac{1}{N_n^2}\sum_{i\in[N_0]}\left\{\sum_{j:E_n^i=i}\varphi(\zeta_n^j)\right\}^2 - \eta_n^N(\varphi)^2\sum_{p=0}^n\frac{1}{N_p} + \mathcal{O}_p(1/N^2)\right], \end{aligned}$$

where the approximation in the final line holds since Proposition 1 implies $\eta_n^N(\varphi)^2$ converges almost surely to $\eta_n(\varphi)^2$. Also using the fact that by Theorem 1, $NV_n^N(\varphi)$ converges in probability to a finite constant, we obtain

$$\begin{aligned} NV_n^N(\varphi) &= NV_n^N(\varphi)\prod_{p=0}^n\left(1-\frac{1}{N_p}\right) + \mathcal{O}_p(1/N) \\ &= \frac{N}{N_n^2}\sum_{i\in[N_0]}\left\{\sum_{j:E_n^i=i}\varphi(\zeta_n^j)\right\}^2 - \eta_n^N(\varphi)^2\sum_{p=0}^n\frac{N}{N_p} + \mathcal{O}_p(1/N) \end{aligned}$$

Taking $\varphi = 1$, $c_p = 1$ and noting $\sum_{i\in[N]}\#_n^i = N$ gives (6). The proof of (7) is similar so the details are omitted.

□

S4 L_r error bounds

As in Remark 2, we define

$$Q_p(x_{p-1}, \mathrm{d}x_p) = G_{p-1}(x_{p-1})M_p(x_{p-1}, \mathrm{d}x_p), \quad p \in \{1,\dots,n\},$$

and $Q_{n,n} = Id$, $Q_{p,n} = Q_{p+1}\cdots Q_n$ for $p \in \{0,\dots,n-1\}$. The following Lemma will be put to multiple uses in our analysis.

Lemma 8. *For any $\varphi \in \mathcal{L}(\mathcal{X})$ and $r \geq 1$,*

$$\sup_{N\geq 1} N^{1/2}E\left\{\left|\gamma_n^N(\varphi)-\gamma_n(\varphi)\right|^r\right\}^{1/r} < \infty, \qquad \sup_{N\geq 1} N^{1/2}E\left\{\left|\eta_n^N(\varphi)-\eta_n(\varphi)\right|^r\right\}^{1/r} < \infty.$$

Proof. Consider the decomposition:

$$\gamma_n^N(\varphi)-\gamma_n(\varphi) = \sum_{p=0}^n \gamma_p^N Q_{p,n}(\varphi) - \gamma_{p-1}^N Q_{p-1,n}(\varphi) = \sum_{p=0}^n \gamma_p^N(1)\frac{1}{N_p}\sum_{i\in[N_p]}\Delta_{p,n}^i,$$

with the convention $\gamma_{-1}^N Q_{-1,n}(\varphi) = M_0 Q_{0,n}(\varphi)$ in the first equality, and

$$\begin{aligned} \Delta_{0,n}^i &= Q_{0,n}(\varphi)(\zeta_p^i) - M_0Q_{0,n}(\varphi), \\ \Delta_{p,n}^i &= Q_{p,n}(\varphi)(\zeta_p^i) - \frac{\eta_{p-1}^N Q_{p-1,n}(\varphi)}{\eta_{p-1}^N(G_{p-1})}, \quad 1\leq p\leq n. \end{aligned}$$

Note that $(\Delta^i_{0,n})_{i\in[N_0]}$ are independent, identically distributed and zero-mean random variables, and for each $p \geq 1$, given $\sigma(\zeta_{0:p-1})$, the $(\Delta^i_{p,n})_{i\in[N_p]}$ are conditionally independent, identically distributed and zero-mean random variables. Moreover, there exists a finite constant say C_n such that $\sup_{N\geq 1}\max_{1\leq p\leq n}\max_{i\in[N_p]}\left|\Delta^i_{p,n}\right| < C_n$ and $\sup_{N\geq 1}\max_{1\leq p\leq n}\gamma^N_p(1) < C_n$. Applying these observations together with the Minkowski and Burkholder–Davis–Gundy inequalities, there exists a finite constant $B_{n,r}$ such that

$$\begin{aligned}\sup_{N\geq 1} N^{1/2} E\left\{\left|\gamma^N_n(\varphi)-\gamma_n(\varphi)\right|^r\right\}^{1/r} &\leq \sup_{N\geq 1} N^{1/2}\sum_{p=0}^{n} E\left\{\left|\gamma^N_p(1)\frac{1}{N_p}\sum_{i\in[N_p]}\Delta^i_{p,n}\right|^r\right\}^{1/r}\\ &\leq B_{n,r}\sup_{N\geq 1} N^{1/2}\sum_{p=0}^{n}\frac{1}{N_p} E\left\{\left|\sqrt{\sum_{i\in[N_p]}(\Delta^i_{p,n})^2}\right|^r\right\}^{1/r} < \infty.\end{aligned}$$

Applying Minkowski's inequality to the decomposition

$$\eta^N_n(\varphi)-\eta_n(\varphi) = \frac{\gamma^N_n(\varphi)}{\gamma^N_n(1)}\left\{\frac{\gamma_n(1)-\gamma^N_n(1)}{\gamma_n(1)}\right\} + \frac{\gamma^N_n(\varphi)-\gamma_n(\varphi)}{\gamma_n(1)}$$

gives

$$\begin{aligned}&\sup_{N\geq 1} N^{1/2} E\left\{\left|\eta^N_n(\varphi)-\eta_n(\varphi)\right|^r\right\}^{1/r}\\ \leq\ &\frac{\sup_x|\varphi(x)|}{\gamma_n(1)}\sup_{N\geq 1} N^{1/2}E\left\{\left|\gamma^N_n(1)-\gamma_n(1)\right|^r\right\}^{1/r} + \frac{1}{\gamma_n(1)}\sup_{N\geq 1} N^{1/2}E\left\{\left|\gamma^N_n(\varphi)-\gamma_n(\varphi)\right|^r\right\}^{1/r},\end{aligned}$$

and the result follows since both terms on the right hand side are finite from the previous bound. □

S5 Proofs of Lemmas 1–3 and Proposition 1

S5.1 Conditional particle filters and proof of Lemma 1

We define $M^N_0(\mathrm{d}z_0) = \prod_{i\in[N_0]} M_0(\mathrm{d}z^i_0)$, and

$$M^N_p(z_{p-1}; a_{p-1}, \mathrm{d}z_p) = \prod_{i\in[N_p]}\frac{G_{p-1}(z^{a^i_{p-1}}_{p-1})M_p(z^{a^i_{p-1}}_{p-1}, z^i_p)}{\sum_{j\in[N_{p-1}]}G_{p-1}(z^j_{p-1})}, \qquad p\geq 1.$$

The probability measure associated with the particle system in Algorithm 1 is specified by

$$P^N(a,\mathrm{d}z) = M^N_0(\mathrm{d}z_0)\prod_{p=1}^{n} M^N_p(z_{p-1}; a_{p-1}, \mathrm{d}z_p).$$

We also define $G^N_p(z_p) = \frac{1}{N_p}\sum_{i\in[N_p]} G_p(z^i_p)$ for $p\in\{0,\ldots,n\}$. Let

$$Q^N_1(k,a,\mathrm{d}z) = P^N(a,\mathrm{d}z)C_1(a,z;k),$$

which specifies the probability measure associated with the random variables (K^1, A, ζ) obtained by simulating the particle system using Algorithm 1 and selecting K^1as described in Section 3.1.

We now introduce the conditional particle filter construction of Andrieu et al. [2010]. Let $-k_p$ denote the set $[N_p]\setminus\{k_p\}$. We define $z^{-k_p}_p = (z^1_p,\ldots,z^{k-1}_p,z^{k+1}_p,\ldots,z^N_p)$, $z^k = (z^{k_0}_0,\ldots,z^{k_n}_n)$ and $z^{-k} = (z^{-k_0}_0,\ldots,z^{-k_n}_n)$, only for the purpose of analysis. We define a variant of M^N_p in which one ancestor index and particle is excluded

$$\bar{M}^N_{p,k_p}(z_{p-1}; a^{-k_p}_{p-1}, \mathrm{d}z^{-k_p}_p) = \prod_{i\in[N_p]\setminus\{k_p\}}\frac{G_{p-1}(z^{a^i_{p-1}}_{p-1})M_p(z^{a^i_{p-1}}_{p-1}, \mathrm{d}z^i_p)}{\sum_{j\in[N_{p-1}]}G_{p-1}(z^j_{p-1})},$$

with $\bar{M}_{0,k_0}^N(\mathrm{d}z_0^{-k_0}) = \prod_{i\in[N_0]\setminus\{k_0\}} M_0(\mathrm{d}z_0^i)$. With a fixed reference path z^k in position k, we define the conditional particle filter to be a Markov kernel defined by

$$\bar{P}_1^N(k, z^k; a, \mathrm{d}z^{-k}) = \bar{M}_{0,k_0}^N(\mathrm{d}z_0^{-k_0}) \prod_{p=1}^n \left\{ \bar{M}_{p,k_p}^N(z_{p-1}; a_{p-1}^{-k_p}, \mathrm{d}z_p^{-k_p}) \mathbb{I}\left(k_{p-1} = a_{p-1}^{k_p}\right) \right\}.$$

This specifies a particular distribution for the particle system excluding z^k, and the ancestor indices conditional upon k and z^k. We also define the Feynman–Kac measure on the path space

$$\underline{\gamma}_n(A) = \int_A M_0(\mathrm{d}x_0) \prod_{p=1}^n G_{p-1}(x_{p-1}) M_p(x_{p-1}, \mathrm{d}x_p), \qquad A \in \mathcal{X}^{\otimes n+1}.$$

Finally,

$$\bar{Q}_1^N(k, a, \mathrm{d}z) = \frac{\mathbb{I}\left(k \in [N_{0:n}]\right)}{|[N_{0:n}]|} \frac{\underline{\gamma}_n(\mathrm{d}z^k)}{\gamma_n(1)} \bar{P}_1^N(k, z^k; \mathrm{d}z^{-k}, a).$$

specifies the probability measure associated with an alternative distribution for (K^1, A, ζ), where K^1 is first sampled uniformly from $[N_{0:n}]$, then $\zeta^{K^1} \sim \underline{\gamma}_n(\cdot)/\gamma_n(1)$ and finally $(A, \zeta^{-K^1}) \sim \bar{P}_1^N(K^1, \zeta^{K^1}; \cdot)$. We denote by $\bar{E}_1$ expectations with respect to the law of this alternative process.

Proof of Lemma 1. The second equality in the statement of the lemma follows from the first since

$$E\left\{\gamma_n^N(\varphi)\right\} = E\left\{\gamma_n^N(1) \frac{1}{N_n} \sum_{i\in[N_n]} \varphi(\zeta_n^i)\right\} = E\left\{\gamma_n^N(1)\varphi(\zeta_n^{K_n^1})\right\} = \gamma_n(\varphi).$$

To establish the first equality in the statement of the lemma, it suffices to show that

$$\left\{\prod_{p=0}^{n-1} G_p^N(z_p)\right\} Q_1^N(k, a, \mathrm{d}z) = \gamma_n(1) \bar{Q}_1^N(k, a, \mathrm{d}z), \tag{S4}$$

since it then follows that $E\left\{\gamma_n^N(1)\varphi(\zeta_n^{K_n^1})\right\} = \bar{E}_1\left\{\gamma_n(1)\varphi(\zeta_n^{K_n^1})\right\} = \gamma_n(\varphi)$. We observe that for any $k \in [N_p]$,

$$M_p^N(z_{p-1}; a_{p-1}, \mathrm{d}z_p) = \frac{G_{p-1}(z_{p-1}^{a_{p-1}^k}) M_p(z_{p-1}^{a_{p-1}^k}, z_p^k)}{\sum_{j\in[N_{p-1}]} G_{p-1}(z_{p-1}^j)} \bar{M}_{p,k}^N(z_{p-1}; a_{p-1}^{-k}, \mathrm{d}z_p^{-k}).$$

Hence,

$$\begin{aligned}
&\left\{\prod_{p=0}^{n-1} G_p^N(z_p)\right\} Q_1^N(k, a, \mathrm{d}z) = \left\{\prod_{p=0}^{n-1} G_p^N(z_p)\right\} P^N(a, \mathrm{d}z) C_1(a, z; k) \\
= \; & \frac{1}{N_n} \left\{\prod_{p=0}^{n-1} G_p^N(z_p)\right\} M_0^N(\mathrm{d}z_0) \prod_{p=1}^n M_p^N(z_{p-1}; a_{p-1}, \mathrm{d}z_p) \mathbb{I}\left(k_{p-1} = a_{p-1}^{k_p}\right) \\
= \; & \frac{1}{N_n} \left\{\prod_{p=0}^{n-1} G_p^N(z_p)\right\} \left\{ \bar{M}_{0,k_0}^N(\mathrm{d}z_0^{-k_0}) \prod_{p=1}^n \bar{M}_{p,k_p}^N(z_{p-1}; a_{p-1}, \mathrm{d}z_p^{-k_p}) \mathbb{I}\left(k_{p-1} = a_{p-1}^{k_p}\right) \right\} \\
& \cdot M_0(\mathrm{d}z_0^{k_0}) \prod_{p=1}^n \frac{G_{p-1}(z_{p-1}^{k_{p-1}}) M_p(z_{p-1}^{k_{p-1}}, \mathrm{d}z_p^{k_p})}{N_{p-1} G_{p-1}(z_p)} \\
= \; & \frac{1}{|[N_{0:n}]|} \left\{ \bar{M}_{0,k_0}^N(\mathrm{d}z_0^{-k_0}) \prod_{p=1}^n \bar{M}_{p,k_p}^N(z_{p-1}; a_{p-1}, \mathrm{d}z_p^{-k_p}) \mathbb{I}\left(k_{p-1} = a_{p-1}^{k_p}\right) \right\} \\
& \cdot M_0(\mathrm{d}z_0^{k_0}) \prod_{p=1}^n G_{p-1}(z_{p-1}^{k_{p-1}}) M_p(z_{p-1}^{k_{p-1}}, \mathrm{d}z_p^{k_p}) \\
= \; & \frac{1}{|[N_{0:n}]|} \bar{P}_1^N(k, z^k; a, \mathrm{d}z^{-k}) \underline{\gamma}_n(\mathrm{d}z^k) = \gamma_n(1) \bar{Q}_1^N(k, a, \mathrm{d}z).
\end{aligned}$$

□

S5.2 Doubly conditional particle filters and proof of Lemma 2

The structure of the proof of Lemma 2 is very similar to the structure of the proof of Lemma 1. Let

$$Q_2^N(k^{1:2}, a, \mathrm{d}z) = P^N(a, \mathrm{d}z)C(a, z; k^{1:2}),$$

which specifies the probability measure associated with the random variables $(K^{1:2}, A, \zeta)$ obtained by simulating the particle system using Algorithm 1 and selecting (K^1, K^2) as described in Section 3.1. We define $\bar{M}^N_{0,k_0^{1:2}}(\mathrm{d}z_0^{-k_0^{1:2}}) = \prod_{i \in [N_0]\setminus\{k_0^{1:2}\}} M_0(\mathrm{d}z_0^i)$,

$$\bar{M}^N_{p,k_p^{1:2}}(z_{p-1}; a_{p-1}^{-k_p^{1:2}}, \mathrm{d}z_p^{-k_p^{1:2}}) = \prod_{i \in [N_p]\setminus\{k_p^{1:2}\}} \frac{G_{p-1}(z_{p-1}^{a_{p-1}^i})M_p(z_{p-1}^{a_{p-1}^i}, \mathrm{d}z_p^i)}{\sum_{j \in [N_{p-1}]}^N G_{p-1}(z_{p-1}^j)},$$

and the Markov kernel associated with a doubly conditional particle filter as

$$\begin{aligned}\bar{P}_2^N(k^{1:2}, z^{k^{1:2}}; a, \mathrm{d}z^{-k^{1:2}}) &= \bar{M}^N_{0,k_0^{1:2}}(\mathrm{d}z_0^{-k_0^{1:2}}) \left\{ \prod_{p=1}^n \bar{M}^N_{p,k_p^{1:2}}(z_{p-1}; a_{p-1}^{-k_p^{1:2}}, \mathrm{d}z_p^{-k_p^{1:2}}) \right\} \\ &\quad \cdot \prod_{p=1}^n \mathbb{I}(k_{p-1}^1 = a_{p-1}^{k_p^1}) \left\{ \mathbb{I}\left(k_p^2 \neq k_p^1, k_{p-1}^2 = a_{p-1}^{k_p^2}\right) + \mathbb{I}\left(k_p^2 = k_p^1\right) \right\}.\end{aligned}$$

A doubly conditional particle filter was also used in Andrieu et al. [2016], but for a different purpose. We also define the path space counterpart for each μ_b, $b \in B_n$, by

$$\underline{\mu}_b(A) = \int_A \tilde{M}_0^{b_0}(\mathrm{d}x_0^{1:2}) \prod_{p=1}^n \tilde{G}_{p-1}(x_{p-1}^{1:2})\tilde{M}_p^{b_p}(x_{p-1}^{1:2}, \mathrm{d}x_p^{1:2}), \qquad A \in \mathcal{X}^{\otimes 2n+2},$$

Finally, we define

$$\bar{Q}_2^N(k^{1:2}, a, \mathrm{d}z) = \frac{\mathbb{I}\left(k^{1:2} \in [N_{0:n}]^2\right)}{|[N_{0:n}]|^2} \frac{\underline{\mu}_{\phi(k^{1:2})}(\mathrm{d}z^{k^{1:2}})}{\mu_{\phi(k^{1:2})}(1)} \bar{P}_2^N(k^{1:2}, z^{k^{1:2}}; a, \mathrm{d}z^{-k^{1:2}}),$$

where $\phi : [N_{0:n}]^2 \to B_n$ maps (k^1, k^2) to the unique $b \in B_n$ such that $(k^1, k^2) \in \mathcal{I}(b)$. $\bar{Q}_2^N$ specifies the probability measure associated with an alternative distribution for $(K^{1:2}, A, \zeta)$, where $K^{1:2}$ is first sampled uniformly from $[N_{0:n}]^2$, then $\zeta^{K^{1:2}} \sim \underline{\mu}_{\phi(K^{1:2})}(\cdot)/\mu_{\phi(K^{1:2})}(1)$ and finally $(A, \zeta^{-K^{1:2}}) \sim \bar{P}_2^N(K^{1:2}, \zeta^{K^{1:2}}; \cdot)$. We denote by $\bar{E}_2$ expectations with respect to the law of this alternative process.

Proof of Lemma 2. The proof of (8)⇒(9) is relatively straightforward so we present that first:

$$\begin{aligned}E\left\{\gamma_n^N(\varphi)^2\right\} &= E\left[\gamma_n^N(1)^2 \left\{\frac{1}{N_n}\sum_{i \in [N_n]} \varphi(\zeta_n^i)\right\}^2\right] \\ &= E\left\{\gamma_n^N(1)^2 \frac{1}{N_n^2} \sum_{i,j \in [N_n]} \varphi(\zeta_n^i)\varphi(\zeta_n^j)\right\} \\ &= E\left\{\gamma_n^N(1)^2 \varphi(\zeta_n^{K_n^1})\varphi(\zeta_n^{K_n^2})\right\} \\ &= E\left[\sum_{b \in B_n} \mathbb{I}\left\{(K^1, K^2) \in \mathcal{I}(b)\right\} \gamma_n^N(1)^2 \varphi(\zeta_n^{K_n^1})\varphi(\zeta_n^{K_n^2})\right] \\ &= \sum_{b \in B_n} E\left[\mathbb{I}\left\{(K^1, K^2) \in \mathcal{I}(b)\right\} \gamma_n^N(1)^2 \varphi(\zeta_n^{K_n^1})\varphi(\zeta_n^{K_n^2})\right] \\ &= \sum_{b \in B_n} \left\{\prod_{p=0}^n \left(\frac{1}{N_p}\right)^{b_p} \left(1 - \frac{1}{N_p}\right)^{1-b_p}\right\} \mu_b(\varphi^{\otimes 2}),\end{aligned}$$

where the final equality is due to (8). To complete the proof of the Lemma it remains to establish (8). For this it suffices to show that

$$\left\{\prod_{p=0}^{n-1} G_p^N(z_p)\right\}^2 Q_2^N(k^{1:2}, a, \mathrm{d}z) = \mu_{\phi(k^{1:2})}(1)\bar{Q}_2^N(k^{1:2}, a, \mathrm{d}z),$$

since it then follows that for any $b \in B_n$,

$$\begin{aligned} E\left[\mathbb{I}\left\{(K^1, K^2) \in \mathcal{I}(b)\right\}\gamma_n^N(1)^2\varphi(\zeta_n^{K_n^1}, \zeta_n^{K_n^2})\right] &= \bar{E}_2\left[\mathbb{I}\left\{(K^1, K^2) \in \mathcal{I}(b)\right\}\mu_b(1)\varphi(\zeta_n^{K_n^1}, \zeta_n^{K_n^2})\right] \\ &= \left(\frac{1}{|[N_{0:n}]|}\right)^2 \sum_{k^1, k^2 \in \mathcal{I}(b)} \mu_b(\varphi) \\ &= \prod_{p=0}^{n} \left(\frac{1}{N_p}\right)^{b_p}\left(1 - \frac{1}{N_p}\right)^{1-b_p} \mu_b(\varphi), \end{aligned}$$

where the last equality follows from the fact that $|\mathcal{I}(b)| = \prod_{p=0}^{n} N_p (N_p - 1)^{1-b_p}$.

We first note that by application of (S4),

$$\begin{aligned} \left\{\prod_{p=0}^{n-1} G_p^N(z_p)\right\}^2 Q_2^N(k^{1:2}, a, \mathrm{d}z) &= \left\{\prod_{p=0}^{n-1} G_p^N(z_p)\right\}^2 Q_1^N(k^1, a, \mathrm{d}z) C_2(a, z, k^1; k^2) \\ &= \left\{\prod_{p=0}^{n-1} G_p^N(z_p)\right\} \gamma_n(1)\bar{Q}_1^N(k^{1:2}, a, \mathrm{d}z) C_2(a, z, k^1; k^2). \end{aligned}$$

We then observe that for any $k^1, k^2 \in [N_{0:n}]$ and $z^{k^1} \in \mathsf{X}^{n+1}$, we have

$$\begin{aligned} &\bar{P}_1^N(k^1, z^{k^1}; a, \mathrm{d}z^{-k^1}) \prod_{p=1}^{n} \left\{\mathbb{I}\left(k_p^2 \neq k_p^1, k_{p-1}^2 = a_{p-1}^{k_p^2}\right) + \mathbb{I}\left(k_p^2 = k_p^1\right)\right\} \\ = \;& \left\{\bar{M}_{0,k_0^1}^N(\mathrm{d}z_0^{-k_0^1}) \prod_{p=1}^{n} \bar{M}_{p,k_p^1}^N(z_{p-1}; a_{p-1}^{-k_p^1}, \mathrm{d}z_p^{-k_p^1})\right\} \\ &\cdot \prod_{p=1}^{n} \mathbb{I}(k_{p-1}^1 = a_{p-1}^{k_p^1}) \left\{\mathbb{I}\left(k_p^2 \neq k_p^1, k_{p-1}^2 = a_{p-1}^{k_p^2}\right) + \mathbb{I}\left(k_p^2 = k_p^1\right)\right\} \\ = \;& \left\{\bar{M}_{0,k_0^{1:2}}^N(\mathrm{d}z_0^{-k_0^{1:2}}) \prod_{p=1}^{n} \bar{M}_{p,k_p^{1:2}}^N(z_{p-1}; a_{p-1}^{-k_p^{1:2}}, \mathrm{d}z_p^{-k_p^{1:2}})\right\} \\ &\cdot \prod_{p=0}^{n} \left\{\mathbb{I}\left(k_p^2 \neq k_p^1\right) \frac{G_{p-1}(z_{p-1}^{k_{p-1}^2}) M_p(z_{p-1}^{k_{p-1}^2}, \mathrm{d}z_p^{k_p^2})}{N_{p-1} G_{p-1}(z_{p-1})} + \mathbb{I}\left(k_p^2 = k_p^1\right)\right\} \\ &\cdot \prod_{p=1}^{n} \mathbb{I}(k_{p-1}^1 = a_{p-1}^{k_p^1}) \left\{\mathbb{I}\left(k_p^2 \neq k_p^1, k_{p-1}^2 = a_{p-1}^{k_p^2}\right) + \mathbb{I}\left(k_p^2 = k_p^1\right)\right\} \\ = \;& \bar{P}_2^N(k^{1:2}, z^{k^{1:2}}; a, \mathrm{d}z^{-k^{1:2}}) \prod_{p=0}^{n} \left\{\mathbb{I}\left(k_p^2 \neq k_p^1\right) \frac{G_{p-1}(z_{p-1}^{k_{p-1}^2}) M_p(z_{p-1}^{k_{p-1}^2}, \mathrm{d}z_p^{k_p^2})}{N_{p-1} G_{p-1}(z_{p-1})} + \mathbb{I}\left(k_p^2 = k_p^1\right)\right\}. \end{aligned}$$

It follows that

$$\begin{aligned}
& \left\{\prod_{p=0}^{n-1} G_p^N(z_p)\right\} \gamma_n(1)\bar{Q}_1^N(k^{1:2}, a, \mathrm{d}z) C_2(a, z, k^1; k^2) \\
= & \left\{\prod_{p=0}^{n-1} G_p^N(z_p)\right\} \frac{1}{|[N_{0:n}]|} \underline{\gamma}_n(\mathrm{d}z^{k^1}) \bar{P}_1^N(k^1, z^{k^1}; a, \mathrm{d}z^{-k^1}) C_2(a, z, k^1; k^2) \\
= & \frac{1}{|[N_{0:n}]|} \left\{\prod_{p=0}^{n-1} G_p^N(z_p)\right\} \bar{P}_2^N(k^{1:2}, z^{k^{1:2}}; a, \mathrm{d}z^{-k^{1:2}}) \\
& \cdot \underline{\gamma}_n(\mathrm{d}z^{k^1}) \prod_{p=0}^{n} \left\{ \mathbb{I}\left(k_p^2 \neq k_p^1\right) \frac{G_{p-1}(z_{p-1}^{k_{p-1}^2}) M_p(z_{p-1}^{k_{p-1}^2}, \mathrm{d}z_p^{k_p^2})}{N_{p-1} G_{p-1}(z_{p-1})} + \mathbb{I}\left(k_p^2 = k_p^1\right) \right\} \\
& \cdot \frac{1}{N_n} \prod_{p=1}^{n} \left\{ \mathbb{I}\left(k_p^2 \neq k_p^1, k_{p-1}^2 = a_{p-1}^{k_p^2}\right) + \mathbb{I}\left(k_p^2 = k_p^1\right) \frac{G_{p-1}(z_{p-1}^{k_{p-1}^2})}{N_{p-1} G_{p-1}(z_{p-1})} \right\} \\
= & \left(\frac{1}{|[N_{0:n}]|}\right)^2 \left\{ \tilde{M}_0^{b_0}(\mathrm{d}x_0^{k_0^{1:2}}) \prod_{p=1}^{n} \tilde{G}_{p-1}(x_{p-1}^{k_{p-1}^{1:2}}) \tilde{M}_p^{b_p}(x_{p-1}^{k_{p-1}^{1:2}}, \mathrm{d}x_p^{k_p^{1:2}}) \right\} \bar{P}_2^N(k^{1:2}, z^{k^{1:2}}; a, \mathrm{d}z^{-k^{1:2}}) \\
= & \left(\frac{1}{|[N_{0:n}]|}\right)^2 \underline{\mu}_{\phi(k^{1:2})}(\mathrm{d}z^{k^{1:2}}) \bar{P}_2^N(k^{1:2}, z^{k^{1:2}}; a, \mathrm{d}z^{-k^{1:2}}) = \mu_{\phi(k^{1:2})}(1) \bar{Q}_2^N(k^{1:2}, a, \mathrm{d}z).
\end{aligned}$$

□

S5.3 Proofs of Lemma 3 and Proposition 1

Proof of Lemma 3. To obtain the limit of $N\mathrm{var}\left\{\gamma_n^N(\varphi)/\gamma_n(1)\right\}$, first combine the equality $E\{\gamma_n^N(\varphi)\} = \gamma_n(\varphi)$ from Lemma 1 and the expression for $E\{\gamma_n^N(\varphi)^2\}$ in Lemma 2 to give

$$\begin{aligned}
N\mathrm{var}\left\{\gamma_n^N(\varphi)\right\} &= -\gamma_n(\varphi)^2 \sum_{p=0}^{n} \frac{N}{\lceil Nc_p \rceil} + \sum_{p=0}^{n} \mu_{e_p}(\varphi^{\otimes 2}) \frac{N}{\lceil Nc_p \rceil} \prod_{q \neq p} \left(1 - \frac{1}{\lceil Nc_q \rceil}\right) + \mathcal{O}(N^{-1}) \\
&= \sum_{p=0}^{n} \frac{N}{\lceil Nc_p \rceil} \left\{ \mu_{e_p}(\varphi^{\otimes 2}) \prod_{q \neq p} \left(1 - \frac{1}{\lceil Nc_q \rceil}\right) - \mu_{0_n}(\varphi^{\otimes 2}) \right\} + \mathcal{O}(N^{-1}).
\end{aligned}$$

Then divide through by $\gamma_n(1)^2$ and take $N \to \infty$. It remains to verify (11). For the remainder of the proof, denote $\varphi_0 = \varphi - \eta_n(\varphi)$. Observe

$$\eta_n^N(\varphi_0) - \frac{\gamma_n^N(\varphi_0)}{\gamma_n(1)} = \eta_n^N(\varphi_0) \left\{1 - \frac{\gamma_n^N(1)}{\gamma_n(1)}\right\},$$

and so by Cauchy–Schwarz,

$$\begin{aligned}
N^{1/2} E\left\{\left|\eta_n^N(\varphi_0) - \frac{\gamma_n^N(\varphi_0)}{\gamma_n(1)}\right|^2\right\}^{1/2} &= N^{1/2} E\left[\left|\eta_n^N(\varphi_0)\left\{1 - \frac{\gamma_n^N(1)}{\gamma_n(1)}\right\}\right|^2\right]^{1/2} \\
&\leq N^{1/2} E\left\{\left|\eta_n^N(\varphi_0)\right|^4\right\}^{1/4} E\left\{\left|1 - \frac{\gamma_n^N(1)}{\gamma_n(1)}\right|^4\right\}^{1/4} \\
&\to 0, \text{ as } N \to \infty, \qquad \text{(S5)}
\end{aligned}$$

where the convergence to zero is a consequence of Lemma 8. Rearranging Minkowski's inequality gives for any random variables X, Y, $E\left\{(X-Y)^2\right\}^{1/2} \geq \left|E\left(X^2\right)^{1/2} - E\left(Y^2\right)^{1/2}\right|$, so the convergence in (S5) implies

$$\left| N^{1/2} E\left\{\eta_n^N(\varphi_0)^2\right\}^{1/2} - N^{1/2} E\left[\left\{\frac{\gamma_n^N(\varphi_0)}{\gamma_n(1)}\right\}^2\right]^{1/2} \right| \to 0, \text{ as } N \to \infty, \qquad \text{(S6)}$$

so $\lim_{N\to\infty} NE\left\{\eta_n^N(\varphi_0)^2\right\} = \lim_{N\to\infty} NE\left[\left\{\gamma_n^N(\varphi_0)/\gamma_n(1)\right\}^2\right]$. The proof is completed by noting that $E\left[\left\{\gamma_n^N(\varphi_0)/\gamma_n(1)\right\}^2\right] = \operatorname{var}\left\{\gamma_n^N(\varphi_0)/\gamma_n(1)\right\}$. □

Proof of Proposition 1. Part 1. holds by Lemma 1. The almost sure convergence in parts 2. and 3. follows from Lemma 8 and the Borel–Cantelli Lemma. The convergence to the asymptotic variances in parts 2. and 3. holds by Lemma 3.

□

S6 Supporting results and proof of Theorem 2

S6.1 Definitions and supporting lemmas

We first introduce a regularity result on randomly weighted, random measures comprised of pairs of independent and identically distributed particles.

Lemma 9. *For each $N \geq 1$, let $(W^{i,j})_{i,j\in[N]}$ be a collection of possibly dependent non-negative random variables. Assume this sequence of collections of random variables satisfies, for any probability measure ν on $\mathcal{X}$ and bounded $\varphi \in \mathcal{L}(\mathcal{X}^{\otimes 2})$,*

$$\sup_N N^{1/2} E\left\{\left(N^{-2}\sum_{i,j\in[N]} W^{i,j}\varphi(\zeta^i,\zeta^j) - \nu^{\otimes 2}(\varphi)\right)^2\right\}^{1/2} < +\infty,$$

where each $\zeta^i \sim \nu$ independently. Then, with $S = \{(i,j,i',j') \in [N]^4 : i',j' \notin \{i,j\}\}$,

$$\sup_N E\left\{N^{-3}\sum_{(i,j,i',j')\in S^\complement} W^{i,j}W^{i',j'}\right\} < +\infty.$$

Proof. Let $\varphi(x,x') = f(x) + f(x')$, where ν and f are taken such that $\nu(\{x : f(x) = 1\}) = \nu(\{x : f(x) = -1\}) = 1/2$. Since $\nu^{\otimes 2}(\varphi) = 0$, we have

$$\sup_N N^{1/2} E\left\{\left(N^{-2}\sum_{i,j\in[N]} W^{i,j}\varphi(\zeta^{i,j})\right)^2\right\}^{1/2} < +\infty.$$

We observe that

$$\begin{aligned}
&E\left\{\varphi(\zeta^i,\zeta^j)\varphi(\zeta^{i'},\zeta^{j'})\right\}\\
=&E\left\{f(\zeta^i)f(\zeta^{i'})\right\} + E\left\{f(\zeta^i)f(\zeta^{j'})\right\} + E\left\{f(\zeta^j)f(\zeta^{i'})\right\} + E\left\{f(\zeta^j)f(\zeta^{j'})\right\},
\end{aligned}$$

so that in particular if $(i,j,i',j') \in S^\complement$ then $E(\varphi(\zeta^i,\zeta^j)\varphi(\zeta^{i'},\zeta^{j'})) \geq 1$. That is,

$$E\left[\left\{\frac{1}{N^2}\sum_{i,j\in[N]} W^{i,j}\varphi(\zeta^{i,j})\right\}^2\right] \geq E\left\{\frac{1}{N^4}\sum_{(i,j,i',j')\in S^\complement} W^{i,j}W^{i',j'}\right\}.$$

The result then follows from

$$\begin{aligned}
\sup_N E\left\{\frac{1}{N^3}\sum_{(i,j,i',j')\in S^\complement} W^{i,j}W^{i',j'}\right\}^{1/2} &= \sup_N N^{1/2} E\left\{\frac{1}{N^4}\sum_{(i,j,i',j')\in S^\complement} W^{i,j}W^{i',j'}\right\}^{1/2}\\
&\leq \sup_N N^{1/2} E\left\{\left(\frac{1}{N^2}\sum_{i,j} W^{i,j}\varphi(\zeta^{i,j})\right)^2\right\}^{1/2} < +\infty.
\end{aligned}$$

□

We recall the definition of $\mu_{b_{0:p}}$ for $p \in \{0, , \ldots, n\}$ in Section S1, noting that it defines a Feynman–Kac model on $\mathcal{X}^{\otimes 2}$, and define

$$\tilde{Q}_p^{b_p}(x_{p-1}^{1:2}, \mathrm{d}x_p^{1:2}) = \tilde{G}_{p-1}(x_{p-1}^{1:2})\tilde{M}_p^{b_p}(x_{p-1}^{1:2}, \mathrm{d}x_p^{1:2}), \qquad p \geq 1.$$

Our analysis involves an induction argument on the number of time steps in the model defined by b and $(\tilde{M}_p^{b_p}, \tilde{G}_p)_{p\in\{0,\ldots,n\}}$, where we allow $(M_p, G_p)_{p\in\{0,\ldots,n\}}$ to be arbitrary with $G_0, \ldots, G_n$ a sequence of $\mathbb{R}$-valued, strictly positive, upper-bounded functions. We define

$$F_{b_0}^N(i_0, j_0) = \mathbb{I}(b_0 = 0, i_0 \neq j_0)\frac{N_0}{N_0 - 1} + \mathbb{I}(b_0 = 1, i_0 = j_0)N_0,$$

and for $p \geq 1$,

$$\begin{aligned} F_{b_{0:p}}^N(i_p, j_p) \quad &= \quad \mathbb{I}(b_p = 0, i_p \neq j_p)\frac{N_p}{N_p - 1}F_{b_{0:p-1}}^N(A_{p-1}^{i_p}, A_{p-1}^{j_p}) \\ &\quad + \mathbb{I}(b_p = 1, i_p = j_p)N_p \sum_{j_{p-1}=1}^{N_{p-1}} F_{b_{0:p-1}}^N(A_{p-1}^{i_p}, j_{p-1})\frac{G_{p-1}(\zeta_{p-1}^{j_{p-1}})}{\sum_{j=1}^{N_{p-1}} G_{p-1}(\zeta_{p-1}^j)}. \end{aligned} \tag{S7}$$

For each p we have

$$\mu_{b_{0:p}}^N(\varphi) = \gamma_p^N(1)^2 \sum_{i_p,j_p\in[N_p]} \frac{1}{N_p^2}F_{b_{0:p}}^N(i_p, j_p)\varphi(\zeta_p^{i_p,j_p}) = \sum_{i_p,j_p\in[N_p]} \frac{1}{N_p^2}W_p^{i_p,j_p}\delta_{\zeta_p^{i_p,j_p}}, \tag{S8}$$

where $W_p^{i_p,j_p} = \gamma_p^N(1)^2F_{b_{0:p}}^N(i_p, j_p)$. Let $\mathcal{F}_p = \sigma(A_0, \ldots, A_{p-1}, \zeta_0, \ldots, \zeta_p)$, $p \geq 1$ and $\mathcal{F}_0 = \sigma(\zeta_0)$. We show that if $\mu_{b_{0:p-1}}^N(\varphi)$ approximates $\mu_{b_{0:p-1}}(\varphi)$ at a $N^{-1/2}$ rate for any $\varphi \in \mathcal{L}(\mathcal{X}^{\otimes 2})$ then $\mu_{b_{0:p}}^N(\varphi)$ approximates $\mu_{b_{0:p}}(\varphi)$ at a $N^{-1/2}$ rate for any $\varphi \in \mathcal{L}(\mathcal{X}^{\otimes 2})$. The first step is to in the following Lemma that a mean-square error bound for $\mu_{b_{0:p-1}}^N(\varphi)$ implies that the random variables $W_{p-1}^{i,j}$ necessarily satisfy a certain regularity condition.

Lemma 10. *If, for any $(M_q, G_q)_{q\in\{0,\ldots,p-1\}}$ and any $\varphi \in \mathcal{L}(\mathcal{X}^{\otimes 2})$,*

$$\sup_N N^{1/2}E\left[\left\{\mu_{b_{0:p-1}}^N(\varphi) - \mu_{b_{0:p-1}}(\varphi)\right\}^2\right]^{1/2} < \infty,$$

then, with $S_{p-1} = \{(i, j, i', j') \in [N_{p-1}]^4 : i', j' \notin \{i, j\}\}$,

$$\sup_N E\left\{N_{p-1}^{-3} \sum_{(i,j,i',j')\in S_{p-1}^{\complement}} W_{p-1}^{i,j}W_{p-1}^{i',j'}\right\} < +\infty.$$

Proof. Let $M_{p-1}(x, \cdot) = \nu(\cdot)$ for every $x \in \mathsf{X}$, where ν is an arbitrary probability measure. Consider the expression for $\mu_{b_{0:p-1}}^N(\varphi)$ in (S8), and note from (S7) that $W_{p-1}^{i,j}$ is measurable with respect to $\sigma(A_{0:p-2}, \zeta_{0:p-2})$. This allows us to apply Lemma 9 to obtain the result. □

Lemmas 11 and 12 together provide important bounds used in the proof of Proposition 3 below. Their proofs involve mainly tedious manipulations involving properties of multinomial random variables, and can be found in Section S6.4. The analysis that follows makes use of the offspring indices defined by (S3).

Lemma 11. *With $E\left\{\Delta_{p,b_p}^N(i_{p-1}, j_{p-1}) \mid \mathcal{F}_{p-1}\right\} = 0$,*

$$\mu_{b_{0:p}}^N(\varphi) - \mu_{b_{0:p-1}}^N(\tilde{Q}_p^{b_p}(\varphi)) = \sum_{i_{p-1},j_{p-1}\in[N_{p-1}]} \frac{1}{N_{p-1}^2}W_{p-1}^{i_{p-1},j_{p-1}}\Delta_{p,b_p}^N(i_{p-1}, j_{p-1}), \tag{S9}$$

where

$$\Delta_{p,0}^N(i_{p-1}, j_{p-1}) = \left[\frac{\left\{\sum_{j=1}^{N_{p-1}} G_{p-1}(\zeta_p^j)\right\}^2}{N_p(N_p - 1)} \sum_{(i_p,j_p)\in O_{p-1}^{i_{p-1}}\times O_{p-1}^{j_{p-1}}} \mathbb{I}(i_p \neq j_p)\varphi(\zeta_p^{i_p,j_p})\right] - \tilde{Q}_p^{b_p}(\varphi)(\zeta_{p-1}^{i_{p-1},j_{p-1}}),$$

and

$$\Delta_{p,1}^N(i_{p-1},j_{p-1}) = \left\{ G_{p-1}(\zeta_{p-1}^{j_{p-1}}) \frac{\sum_{j=1}^{N_{p-1}} G_{p-1}(\zeta_{p-1}^{j})}{N_p} \sum_{i_p \in O_{p-1}^{i_{p-1}}} \varphi(\zeta_p^{i_p,i_p}) \right\} - \tilde{Q}_p^{b_p}(\varphi)(\zeta_{p-1}^{i_{p-1},j_{p-1}}).$$

Lemma 12. *Let $\varphi \in \mathcal{L}(\mathcal{X}^{\otimes 2})$ be non-negative, and $S_{p-1} := \{(i,j,i',j') \in [N_{p-1}]^4 \, : \, i',j' \notin \{i,j\}\}$. Then,*

1. *For any $(i_{p-1},j_{p-1},i'_{p-1},j'_{p-1}) \in S_{p-1} \cap \mathcal{I}(b_{p-1})^2$,*

$$E\left\{\Delta_{p,b_p}^N(i_{p-1},j_{p-1})\Delta_{p,b_p}^N(i'_{p-1},j'_{p-1}) \mid \mathcal{F}_{p-1}\right\} \leq 0.$$

2. *There exists $C < \infty$ such that*

$$E\left\{\Delta_{p,b_p}^N(i_{p-1},j_{p-1})\Delta_{p,b_p}^N(i'_{p-1},j'_{p-1}) \mid \mathcal{F}_{p-1}\right\} \leq C,$$

   for any $(i_{p-1},j_{p-1},i'_{p-1},j'_{p-1}) \in S_{p-1}^{\complement} \cap \mathcal{I}(b_{p-1})^2$.

S6.2 Proof of the theorem

The following proposition constitutes the inductive step in the proof of Theorem 2.

Proposition 3. *If, for any $(M_q,G_q)_{q\in\{0,\ldots,p-1\}}$ and $\varphi \in \mathcal{L}(\mathcal{X}^{\otimes 2})$,*

$$\sup_N N^{1/2} E\left[\left\{\mu_{b_{0:p-1}}^N(\varphi) - \mu_{b_{0:p-1}}(\varphi)\right\}^2\right]^{1/2} < \infty.$$

Then, for any $(M_q,G_q)_{q\in\{0,\ldots p\}}$ and $\varphi \in \mathcal{L}(\mathcal{X}^{\otimes 2})$,

$$\sup_N N^{1/2} E\left[\left\{\mu_{b_{0:p}}^N(\varphi) - \mu_{b_{0:p}}(\varphi)\right\}^2\right]^{1/2} < \infty.$$

Proof. We decompose φ into its positive and negative parts. That is $\varphi = \varphi_+ - \varphi_-$, where $\varphi_+(x^{1:2}) = \max\{0,\varphi(x^{1:2})\}$ and $\varphi_-(x^{1:2}) = |\min\{0,\varphi(x^{1:2})\}|$. We can therefore write

$$\begin{aligned}
& E\left[\left\{\mu_{b_{0:p}}^N(\varphi) - \mu_{b_{0:p-1}}^N(\tilde{Q}_p^{b_p}(\varphi))\right\}^2\right]^{1/2} \\
= \;& E\left[\left\{\mu_{b_{0:p}}^N(\varphi_+) - \mu_{b_{0:p-1}}^N(\tilde{Q}_p^{b_p}(\varphi_+)) + \mu_{b_{0:p-1}}^N(\tilde{Q}_p^{b_p}(\varphi_-)) - \mu_{b_{0:p}}^N(\varphi_-)\right\}^2\right]^{1/2} \\
\leq \;& E\left[\left\{\mu_{b_{0:p}}^N(\varphi_+) - \mu_{b_{0:p-1}}^N(\tilde{Q}_p^{b_p}(\varphi_+))\right\}^2\right]^{1/2} + E\left[\left\{\mu_{b_{0:p-1}}^N(\tilde{Q}_p^{b_p}(\varphi_-)) - \mu_{b_{0:p}}^N(\varphi_-)\right\}^2\right]^{1/2},
\end{aligned}$$

by application of Minkowski's inequality. We bound the first term. It follows from Lemmas 11 and 12 that

$$\begin{aligned}
& E\left[\left\{\mu_{b_{0:p}}^N(\varphi_+) - \mu_{b_{0:p-1}}^N(\tilde{Q}_p^{b_p}(\varphi_+))\right\}^2\right] = E\left[\left\{\sum_{i,j\in[N_{p-1}]} \frac{1}{N_{p-1}^2} W_{p-1}^{i,j}\Delta_{p,b_p}^N(i,j)\right\}^2\right] \\
= \;& E\left\{\sum_{i,j,i',j'\in[N_{p-1}]} \frac{1}{N_{p-1}^4} W_{p-1}^{i,j}W_{p-1}^{i',j'}\Delta_{p,b_p}^N(i,j)\Delta_{p,b_p}^N(i',j')\right\} \\
= \;& E\left[\sum_{i,j,i',j'\in[N_{p-1}]} \frac{1}{N_{p-1}^4} W_{p-1}^{i,j}W_{p-1}^{i',j'} E\left\{\Delta_{p,b_p}^N(i,j)\Delta_{p,b_p}^N(i',j') \mid \mathcal{F}_{p-1}\right\}\right] \\
\leq \;& CE\left\{\sum_{(i,j,i',j')\in S_{p-1}^{\complement}} \frac{1}{N_{p-1}^4} W_{p-1}^{i,j}W_{p-1}^{i',j'}\right\} \leq \frac{C}{N_{p-1}} E\left\{\sum_{(i,j,i',j')\in S_{p-1}^{\complement}} \frac{1}{N_{p-1}^3} W_{p-1}^{i,j}W_{p-1}^{i',j'}\right\}.
\end{aligned}$$

It then follows from Lemma 10 that

$$\sup_N N^{1/2} E\left[\left\{\mu^N_{b_{0:p}}(\varphi_+) - \mu^N_{b_{0:p-1}}(\tilde{Q}^{b_p}_p(\varphi_+))\right\}^2\right]^{1/2} < +\infty,$$

and an identical argument shows that

$$\sup_N N^{1/2} E\left[\left\{\mu^N_{b_{0:p}}(\varphi_-) - \mu^N_{b_{0:p-1}}(\tilde{Q}^{b_p}_p(\varphi_-))\right\}^2\right]^{1/2} < +\infty,$$

so

$$\sup_N N^{1/2} E\left[\left\{\mu^N_{b_{0:p}}(\varphi) - \mu^N_{b_{0:p-1}}(\tilde{Q}^{b_p}_p(\varphi))\right\}^2\right]^{1/2} < +\infty. \tag{S10}$$

Now, $\varphi \in \mathcal{L}(\mathcal{X}^{\otimes 2})$ implies $\tilde{Q}^{b_p}_p(\varphi) \in \mathcal{L}(\mathcal{X}^{\otimes 2})$ because G_{p-1} is bounded, and since $\mu_{b_{0:p-1}}(\tilde{Q}^{b_p}_p(\varphi)) = \mu_{b_{0:p}}(\varphi)$ we have by the hypothesis in the statement

$$\sup_N N^{1/2} E\left[\left\{\mu^N_{b_{0:p-1}}(\tilde{Q}^{b_p}_p(\varphi)) - \mu_{b_{0:p}}(\varphi)\right\}^2\right]^{1/2} < +\infty. \tag{S11}$$

Therefore, (S10), (S11) and Minkowski's inequality together imply the result. □

Proof of Theorem 2. For part 1. of the theorem, let $C_b = \prod_{p=0}^n (N_p)^{b_p} \{N_p/(N_p-1)\}^{1-b_p}$, which is finite by the assumption that $\min_p N_p \geq 2$. By applying Lemma 2,

$$C_b^{-1} E\left\{\mu^N_b(\varphi)\right\} = E\left\{\gamma^N_n(1)^2 \mathbb{I}\left\{(K^1, K^2) \in \mathcal{I}(b)\right\} \varphi(\zeta_n^{K^1_n}, \zeta_n^{K^2_n})\right\} = C_b^{-1}\mu_b(\varphi).$$

The proof of part 2. of the theorem is by induction on n. In the case $n = 0$ we obtain,

$$\begin{aligned}
\mu^N_{b_0}(\varphi) &= \frac{1}{N_0^2} \sum_{i,j\in[N_0]} W_0^{i,j} \varphi(\zeta_0^{i,j}) = \frac{1}{N_0^2} \sum_{(i,j)\in\mathcal{I}(b_0)} N_0^{b_0} \left(\frac{N_0}{N_0-1}\right)^{1-b_0} \varphi(\zeta_0^{i,j}) \\
&= \sum_{(i,j)\in\mathcal{I}(b_0)} \frac{1}{|\mathcal{I}(b_0)|} \varphi(\zeta_0^{i,j}).
\end{aligned}$$

Let $\bar{\varphi}(x^1, x^2) = \varphi(x^1, x^2) - \mu_{b_0}(\varphi)$, and $||\bar{\varphi}|| := \sup_x |\bar{\varphi}(x)|$. We observe that $E\left[\left\{\mu^N_{b_0}(\varphi) - \mu_{b_0}(\varphi)\right\}^2\right] = E\left\{\mu^N_{b_0}(\bar{\varphi})^2\right\}$. In the case $b_0 = 1$, we have

$$E\left[\left\{\mu^N_{b_0}(\varphi) - \mu_{b_0}(\varphi)\right\}^2\right] = E\left\{\sum_{i\in[N_0]} \frac{1}{N_0^2} \bar{\varphi}(\zeta_0^{i,i})^2\right\} = \frac{1}{N_0} E\left\{\bar{\varphi}(X_0, X_0)^2\right\} \leq \frac{1}{N_0} ||\bar{\varphi}||,$$

so $\sup_N N^{1/2} E\left[\left\{\mu^N_{b_0}(\varphi) - \mu_{b_0}(\varphi)\right\}^2\right]^{1/2} < +\infty$ for any $\varphi \in \mathcal{L}(\mathcal{X}^{\otimes 2})$. In the case $b_0 = 0$, we obtain

$$\begin{aligned}
E\left[\left\{\mu^N_{b_0}(\varphi) - \mu_{b_0}(\varphi)\right\}^2\right] &= E\left\{\sum_{i\neq j\in[N_0]} \sum_{i'\neq j'\in[N_0]} \frac{1}{N_0^2(N_0-1)^2} \bar{\varphi}(\zeta_0^{i,j}) \bar{\varphi}(\zeta_0^{i',j'})\right\} \\
&= E\left\{\sum_{(i,j,i',j')\in S_0^{\complement}\cap\mathcal{I}(b_0)} \frac{1}{N_0^2(N_0-1)^2} \bar{\varphi}(\zeta_0^{i,j}) \bar{\varphi}(\zeta_0^{i',j'})\right\} \\
&\leq ||\bar{\varphi}||^2 \frac{|S_0^{\complement} \cap \mathcal{I}(b_0)|}{N_0^2(N_0-1)^2} = ||\bar{\varphi}||^2 \frac{4N_0(N_0-1)(N_0-2) + 2N_0(N_0-1)}{N_0^2(N_0-1)^2} \\
&\leq ||\bar{\varphi}||^2 \frac{4(N_0-2)+2}{N_0(N_0-1)} \leq \frac{4||\bar{\varphi}||^2}{N_0},
\end{aligned}$$

so $\sup_N N^{1/2} E\left[\left\{\mu^N_{b_0}(\varphi) - \mu_{b_0}(\varphi)\right\}^2\right]^{1/2} < +\infty$ for any $\varphi \in \mathcal{L}(\mathcal{X}^{\otimes 2})$. The result then follows by applying Proposition 3 multiple times. □

S6.3 Properties of multinomial random variables

All the results in Lemmas 13–15 can be obtained, after fairly tedious but straightforward calculations, using the moment generating function of a Multinomial$(n, p_1, \dots, p_k)$ random variable X, $M_X(t) = \left(\sum_{i=1}^k p_i e^{t_i}\right)^n$, and the fact that $E\left(\prod_{j=1}^m X_{i_j}\right) = \frac{\partial^m M_X}{\partial t_{i_1} \cdots \partial t_{i_m}}(0)$.

Lemma 13. *Let $(X_1, \dots, X_k)$ be a* Multinomial$(n, p_1, \dots, p_k)$ *random variable. Then*

1. *For any $i \in [k]$, $E(X_i) = np_i$.*
2. *For distinct $i, j \in [k]$, $E(X_i X_j) = n(n-1)p_i p_j$.*
3. *For any $i \in [k]$, $E\{X_i(X_i - 1)\} = n(n-1)p_i^2$.*

Lemma 14. *Let $(X_1, \dots, X_k)$ be a* Multinomial$(n, p_1, \dots, p_k)$ *random variable. Then*

1. *For distinct $i_1, i_2, i_3, i_4 \in [k]$,*

$$E\left(X_{i_1} X_{i_2} X_{i_3} X_{i_4}\right) = \frac{n!}{(n-4)!} \prod_{j=1}^4 p_{i_j} \leq n^2(n-1)^2 \prod_{j=1}^4 p_{i_j} = E\left(X_{i_1} X_{i_2}\right) E\left(X_{i_3} X_{i_4}\right).$$

2. *For distinct $i, j \in [k]$,*

$$\begin{aligned} E\{X_i(X_i-1)X_j(X_j-1)\} &= n(n-1)(n-2)(n-3)p_i^2 p_j^2 \\ &\leq n^2(n-1)^2 p_i^2 p_j^2 = E\{X_i(X_i-1)\}E\{X_j(X_j-1)\}. \end{aligned}$$

Lemma 15. *Let $(X_1, \dots, X_k)$ be a* Multinomial$(n, p_1, \dots, p_k)$ *random variable. Then*

1. *For any $i \in [k]$, $E\left(X_i^2\right) = n(n-1)p_i^2 + np_i$.*
2. *For any $i \in [k]$,*

$$E\left\{X_i^2(X_i-1)^2\right\} = n(n-1)(n-2)(n-3)p_i^4 + 4n(n-1)(n-2)p_i^3 + 2n(n-1)p_i^2.$$

3. *For distinct $i_1, i_2, i_3 \in [k]$,*

$$E\left\{X_{i_1}^2 X_{i_2} X_{i_3}\right\} = n(n-1)(n-2)(n-3)p_{i_1}^2 p_{i_2} p_{i_3} + n(n-1)(n-2)p_{i_1} p_{i_2} p_{i_3}.$$

4. *For distinct $i_1, i_2 \in [k]$,*

$$E\left(X_{i_1}^2 X_{i_2}^2\right) = n(n-1)(n-2)(n-3)p_{i_1}^2 p_{i_2}^2 + n(n-1)(n-2)(p_{i_1}^2 p_{i_2} + p_{i_1} p_{i_2}^2) + n(n-1)p_{i_1} p_{i_2}.$$

Lemma 16. *Let $(X_1, \dots, X_k)$ be a* Multinomial$(n, p_1, \dots, p_k)$ *random variable where $p_i = S^{-1} g_i$, $k/n \leq c$ and $n > 4$. Then there exists a constant $C < \infty$ such that, with $\bar{g} := \max_{i \in \{1,\dots,k\}} g_i$,*

1. *For any $i \in [k]$, $E\left(S^2 X_i^2 / n^2\right) \leq g_i^2 + C\bar{g}^2$.*
2. *For any $i \in [k]$, $E\left\{S^4 X_i^2 (X_i - 1)^2 / n^4\right\} \leq g_i^4 + C\bar{g}^4$.*
3. *For distinct $i_1, i_2, i_3 \in [k]$, $E\left[S^4 X_{i_1}^2 X_{i_2} X_{i_3} / \left\{n^2(n-1)^2\right\}\right] \leq g_{i_1}^2 g_{i_2} g_{i_3} + C\bar{g}^4$.*
4. *For distinct $i_1, i_2 \in [k]$, $E\left[S^4 X_{i_1}^2 X_{i_2}^2 / \left\{n^2(n-1)^2\right\}\right] \leq g_{i_1}^2 g_{i_2}^2 + C\bar{g}^4$.*

Proof. We use the properties from Lemma 15. For part 1.,

$$E\left(\frac{S^2}{n^2} X_i^2\right) = \frac{S^2}{n^2}\left\{n(n-1)p_i^2 + np_i\right\} \leq g_i^2 + \frac{S}{n} g_i \leq g_i^2 + \frac{k}{n}\bar{g}^2.$$

For part 2.,

$$\begin{aligned}
& E\left\{\frac{S^4}{n^2(n-1)^2} X_i^2 (X_i-1)^2\right\} \\
= \quad & \frac{S^4}{n^2(n-1)^2}\left\{n(n-1)(n-2)(n-3)p_i^4 + 4n(n-1)(n-2)p_i^3 + 2n(n-1)p_i^2\right\} \\
\leq \quad & g_i^4 + \frac{4S(n-2)}{n(n-1)} g_i^3 + \frac{2S^2}{n(n-1)} g_i^2 \leq g_i^4 + \frac{4k}{n}\bar{g}^4 + 2\frac{k^2}{n(n-1)}\bar{g}^4 \leq g_i^4 + 4c\bar{g}^4 + 2c^2\frac{n}{n-1}\bar{g}^4.
\end{aligned}$$

For part 3.,

$$\begin{aligned}
E\left\{\frac{S^4}{n^2(n-1)^2} X_{i_1}^2 X_{i_2} X_{i_3}\right\} \quad = \quad & \frac{S^4}{n^2(n-1)^2}\left\{n(n-1)(n-2)(n-3)p_{i_1}^2 p_{i_2} p_{i_3} + n(n-1)(n-2)p_{i_1}p_{i_2}p_{i_3}\right\} \\
\leq \quad & g_{i_1}^2 g_{i_2} g_{i_3} + \frac{S(n-2)}{n(n-1)} g_{i_1} g_{i_2} g_{i_3} \leq g_{i_1}^2 g_{i_2} g_{i_3} + c\bar{g}^4.
\end{aligned}$$

For part 4.,

$$\begin{aligned}
& E\left\{\frac{S^4}{n^2(n-1)^2} X_{i_1}^2 X_{i_2}^2\right\} \\
= \quad & \frac{S^4}{n^2(n-1)^2}\left\{n(n-1)(n-2)(n-3)p_{i_1}^2 p_{i_2}^2 + n(n-1)(n-2)(p_{i_1}^2 p_{i_2} + p_{i_1} p_{i_2}^2) + n(n-1)p_{i_1}p_{i_2}\right\} \\
\leq \quad & g_{i_1}^2 g_{i_2}^2 + \frac{S(n-2)}{n(n-1)}(g_{i_1}^2 g_{i_2} + g_{i_1} g_{i_2}^2) + \frac{S^2}{n(n-1)} g_{i_1} g_{i_2} \leq g_{i_1}^2 g_{i_2}^2 + c\bar{g}^4 + c^2\frac{n}{n-1}\bar{g}^4.
\end{aligned}$$

The result follows by taking $C = 4c + \frac{4}{3}\cdot 2c^2$ since $\frac{n}{n-1} \leq \frac{4}{3}$. □

Corollary 2. *Assume* $\sup_{x\in\mathsf{X}} G_{p-1}(x) < \infty$ *and* $(i,j,i',j') \in S_{p-1}^{\complement} \cap \mathcal{I}(b_{p-1})^2$. *Then there exists* $C < \infty$ *such that for any* $N_p, N_{p-1} \in \mathbb{N}$ *and* $\zeta_{p-1} \in \mathsf{X}^{N_{p-1}}$,

$$E\left[\frac{\left\{\sum_{k=1}^{N_{p-1}} G_{p-1}(\zeta_{p-1}^k)\right\}^4}{N_p^2(N_p-1)^2} |O_{p-1}^i| \left\{|O_{p-1}^j| - \mathbb{I}(i=j)\right\} |O^{i'}| \left\{|O^{j'}| - \mathbb{I}(i'=j')\right\} \mid \mathcal{F}_{p-1}\right] \leq C,$$

and

$$G_{p-1}(\zeta_{p-1}^j) G_{p-1}(\zeta_{p-1}^{j'}) E\left[\frac{\left\{\sum_{k=1}^{N_{p-1}} G_{p-1}(\zeta_{p-1}^k)\right\}^2}{N_p^2} |O_{p-1}^i||O_{p-1}^{i'}| \mid \mathcal{F}_{p-1}\right] \leq C.$$

Proof. Let $k = N_{p-1}$, $n = N_p$, $S = \sum_{j=1}^{N_{p-1}} G_{p-1}(\zeta_{p-1}^j)$ and $(X_1, \ldots, X_k)$ be a Multinomial$(n, p_1, \ldots, p_k)$ random variable where $p_i = S^{-1} G_{p-1}(\zeta_{p-1}^i)$ and $n > 4$. For the first expression, consider the case $i = j$. Then the expression can be written as

$$E\left\{\frac{S^4}{n^2(n-1)^2} X_i^2 (X_i - 1)^2\right\},$$

and we conclude by combining part 2. of Lemma 16 with $\sup_{x\in\mathsf{X}} G_{p-1}(x) < \infty$. Now consider the case $i \neq j$. Then the expression can be written as either

$$E\left\{\frac{S^4}{n^2(n-1)^2} X_i^2 X_j^2\right\}, \qquad \text{or} \qquad E\left\{\frac{S^4}{n^2(n-1)^2} X_{i_1}^2 X_{i_2} X_{i_3}\right\},$$

and we conclude by combining parts 3. and 4. of Lemma 16 with $\sup_{x\in\mathsf{X}} G_{p-1}(x) < \infty$. The second expression can be bounded by $\left\{\sup_{x\in\mathsf{X}} G_{p-1}(x)\right\}^2 E\left(S^2 X_i^2/n^2\right)$ and we conclude by combining the part 1. of Lemma 16 with $\sup_{x\in\mathsf{X}} G_{p-1}(x) < \infty$. □

S6.4 Expressions and bounds for Δ^N_{p,b_p}

Proof of Lemma 11. We obtain expressions for Δ^N_{p,b_p} for each of the cases $b_p \in \{0,1\}$, making use of (S7). In the case $b_p = 0$, we have

$$\begin{aligned}
&\mu^N_{b_{0:p}}(\varphi) = \gamma^N_p(1)^2 \sum_{i_p,j_p\in[N_p]} \frac{1}{N_p^2} F^N_{b_{0:p}}(i_p,j_p)\varphi(\zeta_p^{i_p,j_p}) \\
&= \gamma^N_{p-1}(1)^2 \sum_{i_{p-1},j_{p-1}\in[N_{p-1}]} \frac{1}{N_{p-1}^2} F^N_{b_{0:p-1}}(i_{p-1},j_{p-1}) \frac{\left\{\sum_{j=1}^{N_{p-1}} G_{p-1}(\zeta^j_{p-1})\right\}^2}{N_p(N_p-1)} \sum_{(i_p,j_p)\in O_{p-1}^{i_{p-1}}\times O_{p-1}^{j_{p-1}}} \mathbb{I}(i_p\neq j_p)\varphi(\zeta_p^{i_p,j_p}) \\
&= \sum_{i_{p-1},j_{p-1}\in[N_{p-1}]} \frac{1}{N_{p-1}^2} W_{p-1}^{i_{p-1},j_{p-1}} \frac{\left\{\sum_{j=1}^{N_{p-1}} G_{p-1}(\zeta^j_{p-1})\right\}^2}{N_p(N_p-1)} \sum_{(i_p,j_p)\in O_{p-1}^{i_{p-1}}\times O_{p-1}^{j_{p-1}}} \mathbb{I}(i_p\neq j_p)\varphi(\zeta_p^{i_p,j_p}),
\end{aligned}$$

from which the expression for $\Delta^N_{p,0}(i_{p-1},j_{p-1})$ follows. We have

$$\begin{aligned}
&E\left[\frac{\left\{\sum_{j=1}^{N_{p-1}} G_{p-1}(\zeta^j_{p-1})\right\}^2}{N_p(N_p-1)} \sum_{(i_p,j_p)\in O_{p-1}^{i_{p-1}}\times O_{p-1}^{j_{p-1}}} \mathbb{I}(i_p\neq j_p)\varphi(\zeta_p^{i_p,j_p}) \mid \mathcal{F}_{p-1}\right] \\
&= E\left[\frac{\left\{\sum_{j=1}^{N_{p-1}} G_{p-1}(\zeta^j_{p-1})\right\}^2}{N_p(N_p-1)} |O_{p-1}^{i_{p-1}}|\left(|O_{p-1}^{j_{p-1}}| - \mathbb{I}\{i_{p-1}=j_{p-1}\}\right) \tilde{M}_p^{b_p}(\varphi)(\zeta_{p-1}^{i_{p-1},j_{p-1}}) \mid \mathcal{F}_{p-1}\right] \\
&= \tilde{G}_{p-1}(\zeta_{p-1}^{i_{p-1},j_{p-1}})\tilde{M}_p^{b_p}(\varphi)(\zeta_{p-1}^{i_{p-1},j_{p-1}}),
\end{aligned}$$

where the last line follows from Lemma 13. Hence $E\left\{\Delta^N_{p,0}(i_{p-1},j_{p-1}) \mid \mathcal{F}_{p-1}\right\} = 0$.

In the case $b_p = 1$, we have

$$\begin{aligned}
&\mu^N_{b_{0:p}}(\varphi) = \gamma^N_p(1)^2 \sum_{i_p,j_p\in[N_p]} \frac{1}{N_p^2} F^N_{b_{0:p}}(i_p,j_p)\varphi(\zeta_p^{i_p,j_p}) \\
&= \gamma^N_p(1)^2 \sum_{i_p\in[N_p]} \frac{1}{N_p^2} F^N_{b_{0:p}}(i_p,i_p)\varphi(\zeta_p^{i_p,i_p}) \\
&= \gamma^N_{p-1}(1)^2 \sum_{i_{p-1},j_{p-1}\in[N_{p-1}]} \frac{1}{N_{p-1}^2} F^N_{b_{0:p-1}}(i_{p-1},j_{p-1}) G_{p-1}(\zeta_{p-1}^{j_{p-1}}) \frac{\sum_{j=1}^{N_{p-1}} G_{p-1}(\zeta^j_{p-1})}{N_p} \sum_{i_p\in O_{p-1}^{i_{p-1}}} \varphi(\zeta_p^{i_p,i_p}) \\
&= \sum_{i_{p-1},j_{p-1}\in[N_{p-1}]} \frac{1}{N_{p-1}^2} W_{p-1}^{i_{p-1},j_{p-1}} G_{p-1}(\zeta_{p-1}^{j_{p-1}}) \frac{\sum_{j=1}^{N_{p-1}} G_{p-1}(\zeta^j_{p-1})}{N_p} \sum_{i_p\in O_{p-1}^{i_{p-1}}} \varphi(\zeta_p^{i_p,i_p}),
\end{aligned}$$

from which the expression for $\Delta^N_{p,1}(i_{p-1},j_{p-1})$ follows. We have

$$\begin{aligned}
&E\left\{G_{p-1}(\zeta_{p-1}^{j_{p-1}}) \frac{\sum_{j=1}^{N_{p-1}} G_{p-1}(\zeta^j_{p-1})}{N_p} \sum_{i_p\in O_{p-1}^{i_{p-1}}} \varphi(\zeta_p^{i_p,i_p}) \mid \mathcal{F}_{p-1}\right\} \\
&= G_{p-1}(\zeta_{p-1}^{j_{p-1}}) E\left\{\frac{\sum_{j=1}^{N_{p-1}} G_{p-1}(\zeta^j_{p-1})}{N_p} |O_{p-1}^{i_{p-1}}| \tilde{M}_p^{b_p}(\varphi)(\zeta_{p-1}^{i_{p-1},j_{p-1}}) \mid \mathcal{F}_{p-1}\right\} \\
&= \tilde{G}_{p-1}(\zeta_{p-1}^{i_{p-1},j_{p-1}})\tilde{M}_p^{b_p}(\varphi)(\zeta_{p-1}^{i_{p-1},j_{p-1}}),
\end{aligned}$$

where the last line follows from Lemma 13. Hence $E\left\{\Delta^N_{p,1}(i_{p-1},j_{p-1}) \mid \mathcal{F}_{p-1}\right\} = 0$. □

Proof of Lemma 12. For the first part, consider first the case $b_p = 0$. Then, since $(i_{p-1}, j_{p-1}, i'_{p-1}, j'_{p-1}) \in S_{p-1} \cap \mathcal{I}(b_{p-1})^2$,

$$
\begin{aligned}
& E\left\{\Delta_{p,0}^N(i_{p-1}, j_{p-1})\Delta_{p,0}^N(i'_{p-1}, j'_{p-1}) \mid \mathcal{F}_{p-1}\right\} \\
= \quad & E\left[\frac{\left\{\sum_{j=1}^{N_{p-1}} G_{p-1}(\zeta_p^j)\right\}^4}{N_p^2(N_p-1)^2} \sum_{(i,j)\in O_{p-1}^{i_{p-1}} \times O_{p-1}^{j_{p-1}}} \mathbb{I}(i \neq j)\varphi(\zeta_p^{i,j}) \sum_{(i',j')\in O_{p-1}^{i'_{p-1}} \times O_{p-1}^{j'_{p-1}}} \mathbb{I}(i' \neq j')\varphi(\zeta_p^{i',j'}) \mid \mathcal{F}_{p-1}\right] \\
& -\tilde{Q}_p^{b_p}(\varphi)(\zeta_{p-1}^{i_{p-1},j_{p-1}})\tilde{Q}_p^{b_p}(\varphi)(\zeta_{p-1}^{i'_{p-1},j'_{p-1}}) \\
= \quad & E\left[\frac{\left\{\sum_{j=1}^{N_{p-1}} G_{p-1}(\zeta_p^j)\right\}^4}{N_p^2(N_p-1)^2} |O_{p-1}^{i_{p-1}}| \left(O_{p-1}^{j_{p-1}} - \mathbb{I}\{i_{p-1} = j_{p-1}\}\right) |O_{p-1}^{i'_{p-1}}| \left(|O_{p-1}^{j'_{p-1}}| - \mathbb{I}\{i'_{p-1} = j'_{p-1}\}\right) \mid \mathcal{F}_{p-1}\right] \\
& \cdot\tilde{M}_p^{b_p}(\varphi)(\zeta_{p-1}^{i_{p-1},j_{p-1}})\tilde{M}_p^{b_p}(\varphi)(\zeta_{p-1}^{i'_{p-1},j'_{p-1}}) - \tilde{Q}_p^{b_p}(\varphi)(\zeta_{p-1}^{i_{p-1},j_{p-1}})\tilde{Q}_p^{b_p}(\varphi)(\zeta_{p-1}^{i'_{p-1},j'_{p-1}}) \\
= \quad & \Bigg\{E\left[\frac{\left\{\sum_{j=1}^{N_{p-1}} G_{p-1}(\zeta_p^j)\right\}^4}{N_p^2(N_p-1)^2} |O_{p-1}^{i_{p-1}}| \left(O_{p-1}^{j_{p-1}} - \mathbb{I}\{i_{p-1} = j_{p-1}\}\right) |O_{p-1}^{i'_{p-1}}| \left(|O_{p-1}^{j'_{p-1}}| - \mathbb{I}\{i'_{p-1} = j'_{p-1}\}\right) \mid \mathcal{F}_{p-1}\right] \\
& -\tilde{G}_{p-1}(\zeta_{p-1}^{i_{p-1},j_{p-1}})\tilde{G}_{p-1}(\zeta_{p-1}^{i'_{p-1},j'_{p-1}})\Bigg\} \cdot \tilde{M}_p^{b_p}(\varphi)(\zeta_{p-1}^{i_{p-1},j_{p-1}})\tilde{M}_p^{b_p}(\varphi)(\zeta_{p-1}^{i'_{p-1},j'_{p-1}}) \\
\leq \quad & 0,
\end{aligned}
$$

where the final inequality follows from properties in Lemma 14. Now, in the case $b_p = 1$, and again because $(i_{p-1}, j_{p-1}, i'_{p-1}, j'_{p-1}) \in S_{p-1} \cap \mathcal{I}(b_{p-1})^2$,

$$
\begin{aligned}
& E\left\{\Delta_{p,1}^N(i_{p-1}, j_{p-1})\Delta_{p,1}^N(i'_{p-1}, j'_{p-1}) \mid \mathcal{F}_{p-1}\right\} \\
= \quad & E\left[G_{p-1}(\zeta_{p-1}^{j_{p-1}})G_{p-1}(\zeta_{p-1}^{j'_{p-1}})\frac{\left[\sum_{j=1}^{N_{p-1}} G_{p-1}(\zeta_{p-1}^j)\right]^2}{N_p^2} \sum_{i\in O_{p-1}^{i_{p-1}}} \varphi(\zeta_p^{i,i}) \sum_{i'\in O_{p-1}^{i'_{p-1}}} \varphi(\zeta_p^{i',i'}) \mid \mathcal{F}_{p-1}\right] \\
& -\tilde{Q}_p^{b_p}(\varphi)(\zeta_{p-1}^{i_{p-1},j_{p-1}})\tilde{Q}_p^{b_p}(\varphi)(\zeta_{p-1}^{i'_{p-1},j'_{p-1}}) \\
= \quad & E\left[\frac{\left\{\sum_{j=1}^{N_{p-1}} G_{p-1}(\zeta_{p-1}^j)\right\}^2}{N_p^2} |O_{p-1}^{i_{p-1}}||O_{p-1}^{i'_{p-1}}| \Big| \mathcal{F}_{p-1}\right] G_{p-1}(\zeta_{p-1}^{j_{p-1}})G_{p-1}(\zeta_{p-1}^{j'_{p-1}}) \\
& \cdot\tilde{M}_p^{b_p}(\varphi)(\zeta_{p-1}^{i_{p-1},j_{p-1}})\tilde{M}_p^{b_p}(\varphi)(\zeta_{p-1}^{i'_{p-1},j'_{p-1}}) - \tilde{Q}_p^{b_p}(\varphi)(\zeta_{p-1}^{i_{p-1},j_{p-1}})\tilde{Q}_p^{b_p}(\varphi)(\zeta_{p-1}^{i'_{p-1},j'_{p-1}}) \\
= \quad & \left(E\left[\frac{\left\{\sum_{j=1}^{N_{p-1}} G_{p-1}(\zeta_{p-1}^j)\right\}^2}{N_p^2} |O_{p-1}^{i_{p-1}}||O_{p-1}^{i'_{p-1}}| \Big| \mathcal{F}_{p-1}\right] - G_{p-1}(\zeta_{p-1}^{i_{p-1}})G_{p-1}(\zeta_{p-1}^{i'_{p-1}})\right) \\
& \cdot G_{p-1}(\zeta_{p-1}^{j_{p-1}})G_{p-1}(\zeta_{p-1}^{j'_{p-1}})\tilde{M}_p^{b_p}(\varphi)(\zeta_{p-1}^{i_{p-1},j_{p-1}})\tilde{M}_p^{b_p}(\varphi)(\zeta_{p-1}^{i'_{p-1},j'_{p-1}}) \\
\leq \quad & 0,
\end{aligned}
$$

where the final inequality follows from properties in Lemma 14. For the second part, let $||\varphi|| = \sup_x \varphi(x)$. In the case $b_p = 0$ we have

$$\begin{aligned}
& E\left[\Delta^N_{p,b_p}(i_{p-1},j_{p-1})\Delta^N_{p,b_p}(i'_{p-1},j'_{p-1}) \mid \mathcal{F}_{p-1}\right] \\
\leq\ & E\left[\frac{\left[\sum_{j=1}^{N_{p-1}} G_{p-1}(\zeta_p^j)\right]^4}{N_p^2(N_p-1)^2} \sum_{(i,j)\in O_{p-1}^{i_{p-1}}\times O_{p-1}^{j_{p-1}}} \mathbb{I}(i\neq j)\varphi(\zeta_p^{i,j}) \sum_{(i',j')\in O_{p-1}^{i'_{p-1}}\times O_{p-1}^{j'_{p-1}}} \mathbb{I}(i'\neq j')\varphi(\zeta_p^{i',j'})\middle|\mathcal{F}_{p-1}\right] \\
\leq\ & ||\varphi||^2 E\left[\frac{\left[\sum_{j=1}^{N_{p-1}} G_{p-1}(\zeta_p^j)\right]^4}{N_p^2(N_p-1)^2}|O_{p-1}^{i_{p-1}}|\left(|O_{p-1}^{j_{p-1}}|-\mathbb{I}\{i_{p-1}=j_{p-1}\}\right)|O_{p-1}^{i'_{p-1}}|\left(|O_{p-1}^{j'_{p-1}}|-\mathbb{I}\{i'_{p-1}=j'_{p-1}\}\right)\middle|\mathcal{F}_{p-1}\right] \\
\leq\ & C||\varphi||^2,
\end{aligned}$$

by applying Corollary 2 to obtain the last inequality. Similarly, in the case $b_p = 1$ we have

$$\begin{aligned}
& E\left[\Delta^N_{p,b_p}(i_{p-1},j_{p-1})\Delta^N_{p,b_p}(i'_{p-1},j'_{p-1}) \mid \mathcal{F}_{p-1}\right] \\
\leq\ & E\left[G_{p-1}(\zeta_{p-1}^{j_{p-1}})G_{p-1}(\zeta_{p-1}^{j'_{p-1}})\frac{\left[\sum_{j=1}^{N_{p-1}} G_{p-1}(\zeta_{p-1}^j)\right]^2}{N_p^2}\sum_{i\in O_{p-1}^{i_{p-1}}}\varphi(\zeta_p^{i,i})\sum_{i'\in O_{p-1}^{i'_{p-1}}}\varphi(\zeta_p^{i',i'}) \mid \mathcal{F}_{p-1}\right] \\
\leq\ & ||\varphi||^2 G_{p-1}(\zeta_{p-1}^{j_{p-1}})G_{p-1}(\zeta_{p-1}^{j'_{p-1}})E\left[\frac{\left[\sum_{j=1}^{N_{p-1}} G_{p-1}(\zeta_{p-1}^j)\right]^2}{N_p^2}|O_{p-1}^{i_{p-1}}||O_{p-1}^{i'_{p-1}}| \mid \mathcal{F}_{p-1}\right] \\
\leq\ & C||\varphi||^2,
\end{aligned}$$

by applying Corollary 2 to obtain the last inequality. □

S7 Proofs for Section 5

Lemma 17. *For any $\varphi \in \mathcal{L}(\mathcal{X})$ and $r \geq 1$,*

$$\sup_{N\geq 1} N^{1/2} E\left\{\left|\hat{\gamma}_n^N(\varphi) - \hat{\gamma}_n(\varphi)\right|^r\right\}^{1/r} < \infty, \qquad \sup_{N\geq 1} N^{1/2} E\left\{\left|\hat{\eta}_n^N(\varphi) - \hat{\eta}_n(\varphi)\right|^r\right\}^{1/r} < \infty.$$

Proof. Let $\hat{\varphi}(x) = G_n(x)\varphi(x)$. Since $\hat{\gamma}_n(\varphi) = \gamma_n(\hat{\varphi})$, Lemma 8 provides the first bound. The second bound follows from the first bound and Minkowski's inequality by an essentially identical line of arguments as in the proof of Lemma 8. □

Proof of Proposition 2. The almost sure convergence in both parts follows from Lemma 17 and the Borel–Cantelli Lemma. To obtain the expression for $\hat{\sigma}_n^2(\varphi)$, we have

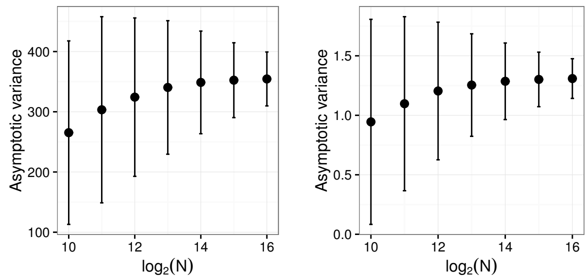
$$\lim_{N\to\infty} N\mathrm{var}\left\{\hat{\gamma}_n^N(\varphi)\right\} = \lim_{N\to\infty} N\mathrm{var}\left\{\gamma_n^N(\hat{\varphi})\right\} = \gamma_n(1)^2\sum_{p=0}^n \frac{v_{p,n}(\hat{\varphi})}{c_p} = \hat{\gamma}_n(1)^2\sum_{p=0}^n \frac{\hat{v}_{p,n}(\varphi)}{c_p},$$

and the result is obtained by dividing by $\hat{\gamma}_n(1)^2$. The expression for $\lim_{N\to\infty} NE\left[\left\{\hat{\eta}_n^N(\varphi) - \hat{\eta}_n(\varphi)\right\}^2\right]$ follows by combining this with an essentially identical line of arguments as in the proof of Lemma 3. □

Proof of Theorem 4. The results follow from Theorems 1 and 3. For the first part,

$$E\left\{\hat{\gamma}_n^N(1)^2\hat{V}_n^N(\varphi)\right\} = E\left\{\gamma_n^N(1)^2 V_n^N(\hat{\varphi})\right\} = \mathrm{var}\left\{\gamma_n^N(\hat{\varphi})\right\} = \mathrm{var}\left\{\hat{\gamma}_n^N(\varphi)\right\}.$$

For the remainder of the proof, $\to$ denotes convergence in probability. For the second part, since $\hat{\sigma}_n^2(\varphi) = \sigma_n^2(\hat{\varphi})/\eta_n(G_n)^2$, it follows that $N\hat{V}_n^N(\varphi) = NV_n^N(\hat{\varphi})/\eta_n^N(G_n)^2 \to \hat{\sigma}_n^2(\varphi)$ since $NV_n^N(\hat{\varphi}) \to \sigma_n^2(\hat{\varphi})$ and $\eta_n^N(G_n)^2 \to \eta_n(G_n)^2$. The third part holds by the same reasoning as in the proof of Theorem 1. □



(a) $\varphi \equiv 1$ (b) $\varphi = Id - \hat{\eta}_n^N(Id)$

Figure 9: Estimated asymptotic variances $N\hat{V}_n(\varphi)$ (blue dots and error bars for the mean $\pm$ one standard deviation) against $\log_2 N$ for the stochastic volatility example.

Proof of Theorem 5. For part 1.,

$$E\left\{\hat{\gamma}_n^N(1)^2\hat{v}_{p,n}^N(\varphi)\right\} = E\left\{\gamma_n^N(1)^2 v_{p,n}^N(\hat{\varphi})\right\} = \gamma_n(1)^2 v_{p,n}(\hat{\varphi}) = \hat{\gamma}_n(1)^2\hat{v}_{p,n}(\varphi).$$

For the remainder of the proof, $\to$ denotes convergence in probability. For part 2., we have $\hat{v}_{p,n}^N(\varphi) = v_{p,n}^N(\hat{\varphi})/\eta_n^N(G_n)^2 \to \hat{v}_{p,n}(\varphi)$ and letting $f = \varphi - \hat{\eta}_n(\varphi)$ we obtain $\hat{v}_{p,n}^N(\varphi - \hat{\eta}_n^N(\varphi)) \to v_{p,n}(\hat{f})/\eta_n(G_n)^2 = \hat{v}_{p,n}(f)$. Part 3. follows from parts 1. and 2. □

S8 Supplementary figures

References

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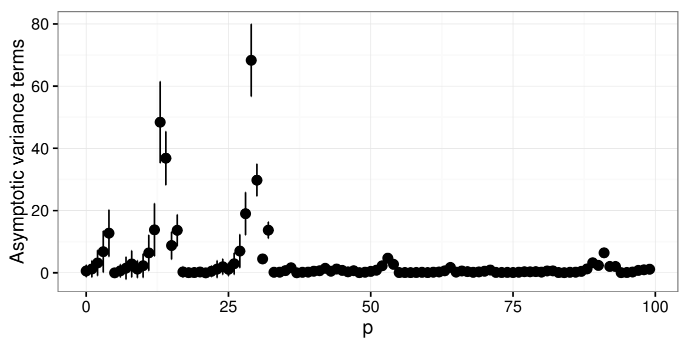
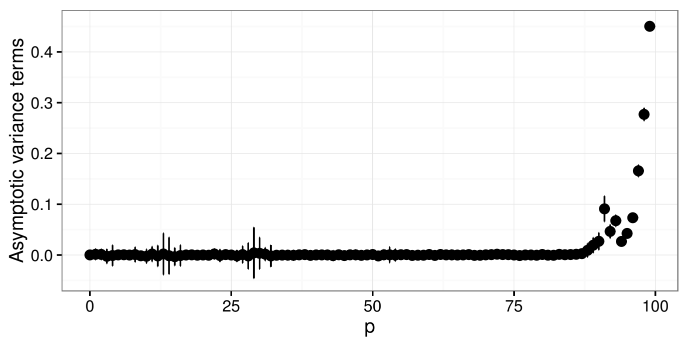
C. Andrieu, A. Lee, and M. Vihola. Uniform ergodicity of the iterated conditional SMC and geometric ergodicity of particle Gibbs samplers. *Bernoulli*, 2016. , to appear.

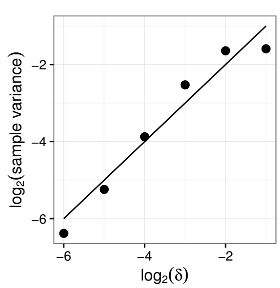


(a) $\varphi \equiv 1$

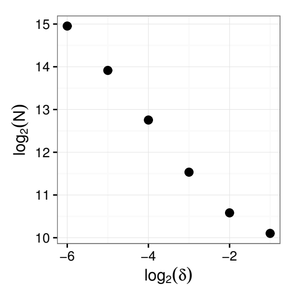


(b) $\varphi = Id - \hat{\eta}_n(Id)$

Figure 10: Plot of $\hat{v}_{p,n}^N(\varphi)$ (blue dots and error bars for the mean $\pm$ one standard deviation) at each $p \in \{0, \ldots, n\}$ in the stochastic volatility example, with $N = 10^5$.

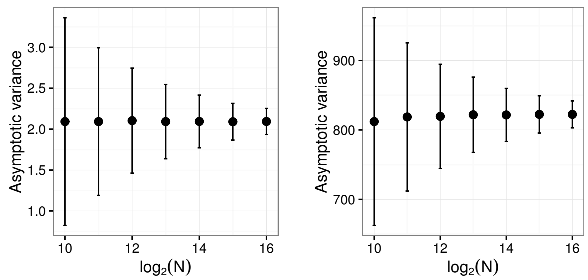


(a)



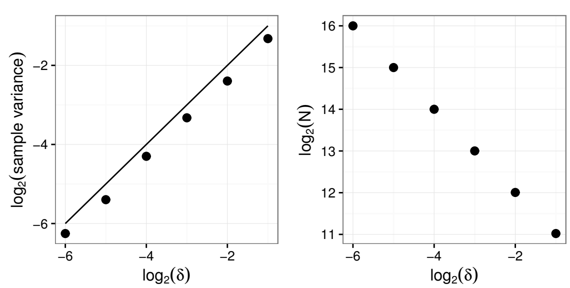
(b)

Figure 11: Plots for the simple adaptive N particle filter estimates of $\hat{\gamma}_n(1)$ for the stochastic volatility example. Figure (a) plots the base 2 logarithm of the empirical variance of $\hat{\gamma}_n^N(1)/\hat{\gamma}_n(1)$ against $\log_2 \delta$, with the straight line $y = x$. Figure (b) plots $\log_2 N$ against $\log_2 \delta$, where N is the average number of particles used by the final particle filter.



(a) $\varphi \equiv 1$ (b) $\varphi = Id - \eta_n^N(Id)$

Figure 12: Estimated asymptotic variances $NV_n^N(\varphi)$ (blue dots and error bars for the mean ± one standard deviation) against $\log_2 N$ for the SMC sampler example.



(a) (b)

Figure 13: Plots for the simple adaptive N particle filter estimates of $\eta_n(Id)$ for the SMC sampler example. Figure (a) plots the base 2 logarithm of the squared L_2 error of $\eta_n^N(Id)$ against $\log_2 \delta$, with the straight line $y = x$. Figure (b) plots $\log_2 N$ against $\log_2 \delta$, where N is the average number of particles used by the final particle filter.